CHAPTER 8

Rational Diverse Beliefs and Market Volatility

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Abstract

This chapter explores theories that explain market dynamics by using rational but diverse beliefs and employing mechanisms of endogenous amplification. Noisy Rational Expectations Asset-Pricing Theory (in which belief diversity arises from diverse private information) does not offer a satisfactory paradigm, because increased idiosyncratic private information *reduces* market volatility. We then focus on models of diverse beliefs *without* private information, whereby economic agents do not know the structure of the complex economy but infer empirical probability from data. A belief is a model of *deviations* from this empirical distribution. It is shown that in a world of diverse beliefs, *to be rational, a belief must fluctuate around the empirical frequencies*, generating endogenous amplification. *Market belief*, which is the distribution of individual beliefs, is then observable via sampling. We explore an explicitly solvable asset-pricing model with diverse beliefs to illustrate the central implications of the theory for market dynamics, the nature of uncertainty and risk premia. *Simulations* are employed to illustrate the ability of the theory to explain the stylized empirical facts. The results offer a unified explanation of the key features of market dynamics, such as excess price volatility, the Equity Premium Puzzle, predictability of asset returns, and stochastic volatility.

Keywords: diverse beliefs, private information, rational beliefs, market beliefs, empirical probability, stable probability, excess volatility, endogenous uncertainty, volume of trade, risk premium

8.1. INTRODUCTION

This chapter explores the role of rational diverse beliefs in explaining market dynamics and volatility. To do that we examine alternative reasons for belief diversity, which is rational, and review different models of market dynamics that incorporate such beliefs. The term *market dynamics* refers to the dynamic characteristics of financial markets, but we focus on dynamic phenomena that attracted attention in the literature. Examples include excess volatility of asset returns, high and time-varying equity risk premia, high volume of trade, and so on. Many examples are termed "anomalies" or "puzzles" because they contradict predictions of *rational expectations equilibria* (in short, REE) with full information. Most studies show that fundamental exogenous factors cannot explain the observed dynamics, leading Paul Samuelson to quip that "the stock market predicted nine of the last five recessions." We aim to explore ideas that explain the four recessions the market predicted but that did not occur. The problem of market volatility and the question of whether asset markets exhibit "excess" volatility relative to fundamental factors have been central in financial economics; thus, the approach explored here addresses major questions and offers useful ideas for advancing the scientific study of free markets, for public stabilization policy, and for practitioners in financial markets.

It is important to note that the idea that allocations and prices are affected by agents' perceptions of the future is rather old. Diverse expectations are central to Thornton's

(1802) views of paper money and financial markets. Expectations are crucial to Keynes (1936). Chapter 12 of *The General Theory* examines the "state of confidence" and the importance of investors' expectations to asset pricing. Expectations are key to the "cumulative movements" in Pigou (see Pigou, 1941, Chapter VI) and constitute the mechanism of deviations from a stationary equilibrium in the Swedish school (e.g., see Myrdal's 1939 views in Myrdal, 1962, Chapter III). Also, "subjective values" based on diverse agents' expectations are cornerstones of Lindahl's (1939) theory of money and capital.

Before turning to recent work, we draw attention to an assumption made in the work reviewed later. It holds that *the distribution of beliefs in the market is an observable variable* that can be deduced from forecast panel data. It would thus be useful to briefly review some available raw data.

In the post-World War II era, large databases on heterogeneous forecasts of various variables have been assembled in Holland, Germany, and Sweden. In the United States, the Survey of Professional Forecasts, reporting *quarterly* forecasts of private forecasters, was started in 1968. It was first conducted by the American Statistical Association's National Bureau but has since been taken over by the Federal Reserve Bank of Philadelphia. Since 1980 the Blue Chip Economic Indicators reports *monthly* forecasts of economic variables by over 50 financial institutions. This service was expanded, under the title of Blue Chip Financial Forecasts, to include forecasts of interest rates and other variables. To illustrate, Table 8.1 reports forecasts of GDP growth and change in GDP deflators in May 2000 for the year 2000. Actual GDP growth rate in 2000 was 4.1% and actual inflation rate was 2.3%. Note that in May 2000, *five months into the year*, large heterogeneity persisted. Also, almost all GDP growth forecasts were wrong! To understand this correlated error, place yourself in May 2000 and make a stationary econometric forecast of GDP growth, but make no special judgment about the unique conditions in May 2000. An example of such a model was developed by Stock and Watson (2001, 2002, 2005). They estimate it by using a combination of diffusion indexes and bivariate VAR forecasts and employing a large number of U.S. time series. In May 2000 the nonjudgmental stationary forecast of GDP growth was *lower than most private forecasts*.

Repeating the experiment over time, we find that the distribution of forecasts fluctuates in two ways. First, it exhibits changes in the *cross-sectional variance* of the forecasts, reflecting changes in degree of disagreement. Second, it exhibits large fluctuations over time *in relation to the stationary forecasts* reflecting correlation in forecasters' views about unusual conditions at the time. Sometimes forecast distributions are below the stationary forecast, whereas in May 2000 the distribution was above the stationary forecast. Observe that the noted large data banks of market forecast distributions are publically available for many variables because forecasters are willing to reveal their forecasts.

We thus note that for any variable, *individual forecasts are correlated and the average market forecast fluctuates around the stationary forecast*. This nonjudgmental forecast is a central yardstick in the work reviewed later. Also, observe that the cross-sectional variance of forecasts fluctuates over time.

Despite the impact of the rational expectations paradigm, belief heterogeneity has been used to explain many phenomena such as asset price volatility, risk premia, volume

TABLE 8.1 May 2000 BLUE Forecasts of Growth and Inflation Rates for the Full Year 2000

May 2000 Forecasted Percent Change in Forecast for 2000	Real GDP Growth	GDP Price Deflator
First Union Corp.	5.3H	2.0
Turning Points (Micrometrics)	5.2	2.1
J. P. Morgan	5.2	2.1
Evans, Carroll & Assoc.	5.1	2.2
Mortgage Bankers Association	5.1	2.1
Goldman Sachs Group, Inc.	5.1	2.1
U.S. Trust Company	5.1	2.0
U.S. Chamber of Commerce	5.1	2.0
Bank of America Corp.	5.1	2.0
Morgan Stanley Dean Witter	5.1	1.9
Wayne Hummer Investments LLC	5.0	2.3
Bank One Corp.	5.0	2.1
Nomura Securities Co.	5.0	1.9
Merrill Lynch	5.0	1.9
Perna Associates	4.9	2.3
National Assn. of Home Builders	4.9	2.1
Macroeconomic Advisers, LLC	4.9	2.1
Prudential Securities, Inc.	4.9	2.0
LaSalle National Bank	4.8	2.3
Conference Board	4.8	2.3
Wells Capital Management	4.8	2.2
DuPont	4.8	2.1
Northern Trust Company	4.8	2.1
Chicago Capital, Inc.	4.8	2.0
Deutsche Bank Securities	4.8	1.8
Chase Securities, Inc.	4.8	1.8
Credit Suisse First Boston	4.8	1.8
Comerica	4.7	2.4
Moody's Investors Service	4.7	2.2
Fannie Mae	4.7	2.0
Federal Express Corp.	4.7	2.0
SOM Economics, Inc.	4.7	1.9
National Assn. of Realtors	4.7	1.9
National City Corporation	4.7	1.9
ClearView Economics	4.7	1.9

(Continued)

TABLE 8.1 (*Continued*)

May 2000 Forecasted Percent Change in Forecast for 2000	Real GDP Growth	GDP Price Deflator
Eggert Economic Enterprises, Inc.	4.6	2.1
WEFA Group	4.6	1.9
Eaton Corporation	4.6	1.9
Bear Stearns & Co., Inc.	4.6	1.2 L
Ford Motor Company	4.5	1.8
Motorola	4.5	1.7
Standard & Poors Corp.	4.5	1.7
UCLA Business Forecasting Project	4.4	2.1
Inforum–University of Maryland	4.4	2.0
Prudential Insurance Co.	4.4	1.9
Weyerhaeuser Company	4.3	2.2
DaimlerChrysler AG	4.3	2.0
Georgia State University	4.2	2.2
Kellner Economic Advisers	4.2	2.0
Econoclast	4.1	2.0
Naroff Economic Advisors	4.0 L	2.5 H

of trade, and money nonneutrality. There are two general theories of rational behavior motivated by the observed diversity. One follows the Harsanyii doctrine, viewing people as Bayesians who hold the same prior probability but with asymmetric private information used in forecasting.

Examples of supporting papers include Grossman and Stiglitz (1980), Phelps (1970), Lucas (1972), Diamond and Verrecchia (1981), Townsend (1978, 1983), Singleton (1987), Brown and Jennings (1989), Grundy and McNichols (1989), Wang (1994), He and Wang (1995), Judd and Bernardo (1996, 2000), Morris and Shin (2002, 2005), Woodford (2003), Hellwig (2002, 2005), Angeletos and Pavan (2006), Angeletos and Werning (2006), Allen, Morris, and Shin (2006), and Hellwig, Mukherji, and Tsyvinski (2006).

An alternative view sees no evidence for the use of private information in forecasts of market prices or economic aggregates. It finds no justification for a common prior and insists that diverse beliefs about state variables are inevitable in a complex world. A sample of papers taking this approach include Harrison and Kreps (1978), Varian (1985, 1989), Harris and Raviv (1993), Kurz (1994, 1996, 1997a, 1997b, 1997c, 2007) Detemple and Murthy (1994), Frankel and Rose (1995), Kandel and Pearson (1995), Cabrales and Hoshi (1996), Kurz and Beltratti (1997), Brock and Hommes (1997, 1998), Kurz and Motolese (2001, 2007), Kurz and Schneider (1996), Kurz and Wu (1996), Kurz, Jin and Motolese (2005a, 2005b), Nielsen (1996, 2003, 2005, 2006), Motolese (2001, 2003), Wu and Guo (2003, 2004), Fan (2006), Acemoglu et al. (2007), Nakata (2007), and Guo and Wu (2007).

We first clarify the differences between these two theories and examine whether they solve the problems outlined. Since the range of issues is wide, we concentrate on rationalized beliefs, and this excludes two types of models. The first type is behavioral finance and noise trading models, in which belief diversity arises from psychological but irrational motives. The second is learning models with common information, typically models of convergence to rational expectations; hence in these models belief diversity is not persistent. Before turning to an evaluation of the difference between these two approaches, it is useful to clarify the standards we set for advocating or rejecting a theory.

Since standard models explain dynamics with exogenous shocks and these are not sufficient to explain the data, to explain excess volatility we search for mechanisms of *endogenous amplification*. In addition, it will become clear that not all belief heterogeneity generates market dynamics. Hence, we ask, What must be the structure of heterogeneity for belief diversity to matter? The data reveal that heterogeneity persists; therefore the two key criteria for effective diversity in any theory are that *it has aggregate effects and that these effects are nonvanishing*. The nonvanishing condition is challenging to models of asymmetric information under rational expectations since information revelation of prices leads back to a common belief, and hence diversity cannot persist. Asymmetric private information in a rational expectations equilibrium is therefore usually supplemented with a "noisy" mechanism to avoid revelation. But then we must ask whether such a mechanism is natural or is it just an artificial construct? Is it testable? Requiring diversity to have persistent aggregate effects implies that heterogeneity by itself is not sufficient and it must be supplemented with dynamic features. To understand the importance of this fact, consider two examples:

1. Beliefs are diverse, randomly and independently distributed over agents with a fixed distribution over time. Here an agent's belief measured by, say, a density over states, changes over time but is randomly determined. The IID distribution causes a cancellation of the effects of beliefs; hence there is a constant, typically small, aggregate effect. Such a distribution implies that diversity is irrelevant.
2. Beliefs are heterogeneous *with a fixed distribution of beliefs so that the belief of each agent is fixed on that distribution*. An agent always has the same superior or inferior information, or else specific agents are always more optimistic or more pessimistic than others. A fixed belief distribution also implies that prices, volume of trade, and risk premia fluctuate only in response to exogenous shocks; hence we are back to a theory, rejected by the data, that publicly observed exogenous shocks are the only cause of fluctuations. Such distributions of beliefs do not generate the desired endogenous amplification to explain excess volatility.

These examples show that the *dynamics* of beliefs over time are essential, and the question is, what is their source? Under asymmetric private information such dynamics could be generated by an exogenous flow of private information, which entails a process of belief updating. How effective or plausible such an assumption is must be carefully weighed, and we discuss it in detail in Section 8.2.3. The situation is different under diverse beliefs with common information, since dynamics and rationality are inherently interrelated. We explain in Section 8.3.1 that the central principle that drives the theory of rational heterogeneous beliefs is that *rational diversity of correlated beliefs without private information implies market volatility.*

The chapter is organized as follows: Section 8.2 reviews the structure of *noisy REE* asset-pricing theory. It shows that although this theory has many useful features, it fails to deliver a consistent theory of financial market dynamics. Indeed, volatility and volume of trade *decline with belief diversity.* Section 8.3 reviews the theory of diverse beliefs with common information. It shows that the theory delivers a consistent and plausible model of endogenous amplification and provides a foundation for understanding market dynamics. Section 8.4 reviews simulation models based on the theory in Section 8.3. It shows that simulations of models with diverse beliefs match the observed data well. Finally, Section 8.5 reviews some open problems.

8.2. CAN MARKET DYNAMICS BE EXPLAINED BY ASYMMETRIC PRIVATE INFORMATION?

The literature on asset pricing in "noisy" REE under asymmetric private information is large, and Brunnermeier (2001) provides a good survey. We discuss it in three stages. In Section 8.2.1 we present a universally used model with exponential utility. In Section 8.2.2 we discuss dynamic versions of the model. In Section 8.2.3 we evaluate the developed ideas.

8.2.1. A General Model of Asset Pricing under Asymmetric Information

The model reviewed here is an adaptation of the short-lived trader model of Brown and Jennings (1989). Similar models were used by Grossman and Stiglitz (1980), Diamond and Verrecchia (1981), Singleton (1987), Grundy and McNichols (1989), Wang (1994), He and Wang (1995), Bacchetta and van Wincoop (2005, 2006), Allen, Morris, and Shin (2006), and others.

There is a unit mass of traders, indexed by the [0, 1] interval and a single aggregate asset with unknown intrinsic unit value V. The economy is static with one period divided into three trading dates (no discounting): At date 1, traders first receive public information y and private signals x^i about the asset value and then they trade. At date 2 they trade again. At date 3 (or end of date 2) uncertainty is resolved, the true liquidation value V is revealed and traders receive it for their holdings. Public information is that V is distributed in accord with $V \sim N(y, \frac{1}{\alpha})$. The private signal x^i about V is $x^i = V + \varepsilon^i$, where ε^i satisfy $\varepsilon^i \sim N(0, \frac{1}{\beta})$ *independently across* i. (α, β) are known. Since these facts are common knowledge, agents know the true unknown value V is "in the market," since by the law of large numbers the mean of all private signals is the true value V. Trader i starts with S^i units of the aggregate asset and can borrow at zero interest rate to finance trading. (D_1^i, D_2^i) are i's demands in the first and second rounds and (p_1, p_2) are market prices in the two rounds. Ending wealth is thus $W^i = S^i p_1 + D_1^i(p_2 - p_1) + D_2^i(V - p_2)$.

All traders are assumed to have the same utility over wealth $u(W^i) = -e^{-(W^i/\tau)}$, and they maximize expected utility. Aggregate supplies (S_1, S_2) of shares, each representing an asset unit, traded in each of the rounds, are *random, unobserved, and independently normally distributed with mean zero*. This noise is crucial to ensure that traders cannot deduce from prices the true value of V. In a noisy REE traders maximize expected utility of final wealth while markets clear after traders deduce from prices all possible information. Indeed, Brown and Jennings (1989) show that equilibrium price at date 1 is of the form

$$p_1 = \kappa_1(\lambda_1 y + \mu_1 V - S_1) \tag{8.1a}$$

and since S_1 is normally distributed, p_1 is also normally distributed. Keep in mind that V and S_1 are unknown, hence Eq. 8.1a shows that prices are not fully revealing.

Since over trading rounds V remain fixed, more rounds of trading generate more price data from which traders deduce added information about V. But with additional supply shocks the inference problem becomes more complicated. That is, at date 2 the price p_2 contains more information about V, but it depends on *two* unobserved noise shocks (S_1, S_2). Hence, the price function is shown to be time dependent and at date 2 takes the form

$$p_2 = \hat{\kappa}_2(\hat{\lambda}_2 y + \hat{\mu}_2 V - S_2 + \psi S_1) \tag{8.1b}$$

Since the realized noise S_1 is not observed, traders condition on the known price p_1 to infer what they can about S_1. They thus use a date 2 price function, which takes an equivalent form

$$p_2 = \kappa_2(\lambda_2 y + \mu_2 V - S_2 + \xi_{21} p_1) \tag{8.1c}$$

By Eq. 8.1a, equivalence means $\kappa_2 = \hat{\kappa}_2, \lambda_2 = (\hat{\lambda}_2 + \lambda_1\psi), \mu_2 = (\hat{\mu}_2 + \mu_1\psi)$ and $\xi_{21} = -\frac{\psi}{\kappa_1}$. Denote by (H_1^i, H_2^i) the information of i in the two rounds. The linearity of the equilibrium price map implies that the payoff is normally distributed. Brown and Jennings (1989) show in their Appendix A that there exist constants (G_1, G_2), determined by the covariance matrix of the model's random variables and assumed by most writers to be the same for all agents, such that the demand functions of i are

$$D_2^i(p_2) = \frac{\tau}{\mathrm{Var}^i(V|H_2^i)}[E^i(V|H_2^i) - p_2] \tag{8.2a}$$

$$D_1^i(p_1) = \frac{\tau}{G_1}[E^i(p_2|H_1^i) - p_1] + \frac{(G_2 - G_1)}{G_1}[E^i(D_2^i|H_1^i)] \tag{8.2b}$$

Most writers assume $\text{Var}^i(V|H_2^i) = \sigma_V^2$ independent of i. The second term in Eq. 8.2b is the "hedging demand" arising from a trader's date 1 perceived risk of price change between date 1 and date 2. The hedging demand in a noisy REE complicates the inference problem and raises equilibrium existence problems. As a result, most writers ignore this demand and study *myopic-investor economies*, where there are only "short-lived" traders who live one period only. They first trade in date 1, gain utility from p_2, and leave the economy. They are replaced by new short-lived traders who receive the information of the first traders but trade in date 2 only and gain utility from the revealed V. None of them have hedging demands. A "long-lived" trader lives through *both* periods, trades in dates 1 and 2, and hence has a hedging demand. For simplicity we follow here the common practice and ignore the second term in Eq. 8.2b. We now average on i, equate to supply and conclude that

$$p_2 = \overline{E}_2(V) - (\frac{1}{\tau})\sigma_V^2(S_1 + S_2), \quad p_1 = \overline{E}_1(p_2) - \frac{G_1}{\tau}S_1 \qquad \textbf{(8.3)}$$

$\overline{E}_2(V)$ is date 2 average market forecast of V and $\overline{E}_1(p_2)$ is average market forecast of p_2. In this case $G_1 = \text{Var}_1^i(p_2)$ and it is assumed this variance is the same for all i. Hence, the proof of Eqs. 8.1a and 8.1b amounts to exhibiting a closed form solution of $\overline{E}_2(V)$ and solving the joint system in Eq. 8.3.

The derivation of Eqs. 8.2a and 8.2b used a *general* form of conditional expectations and required only that prices are normally distributed. It is thus a general solution for any informational structure used in the conditioning and *it does not depend on the private character of information.* Hence, Eqs. 8.2a and 8.2b are applicable to models with diverse beliefs and common information as long as their implied prices are normally distributed. Moreover, differences among theories of diverse beliefs are expressed entirely by differences in their implications to the conditional expectations in Eqs. 8.2a and 8.2b. In the case of asymmetric private information discussed here, Eq. 8.2a shows that D_2^i depend on date 2 expectations, which are updated based on the information deduced from p_2 and p_1. This is different from date 1 information, which consists of a public signal, private signals, and inference from p_1 only. This is why equilibrium price maps are time dependent. Allen et al. (2006) present in their Appendix A computations of the closed-form solution. To get an idea of the inference involved, we briefly review the steps they take.

What does a trader learn in Round 1? Given a prior belief $V \sim N(y, \frac{1}{\alpha})$, trader i observes $p_1 = \kappa_1(\lambda_1 y + \mu_1 V - S_1)$. Since $S_1 \sim N(0, 1/\gamma_1)$, all he infers from date 1 price is that

$$[1/(\kappa_1\mu_1)](p_1 - \kappa_1\lambda_1 y) = V - [S_1/\mu_1] \sim N(V, 1/(\mu_1^2\gamma_1))$$

But now his added piece of information is the private signal $x^i = V + \epsilon^i, \epsilon^i \sim N(0, \frac{1}{\beta})$. Using a standard Bayesian inference from these three sources, his posterior belief becomes

$$E_1^i(V|H_1^i) = \frac{\alpha y + \beta x^i + \mu_1^2\gamma_1 \frac{1}{\kappa_1\mu_1}(p_1 - \kappa_1\lambda_1 y)}{\alpha + \beta + \mu_1^2\gamma_1} = \frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta x^i + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1} \tag{8.4a}$$

with precision

$$\alpha + \beta + \mu_1^2\gamma_1 \tag{8.4b}$$

Averaging Eq. 8.4a over the population, we can see that the average market forecast at date 1 is then

$$\overline{E}_1(V|H_1) = \frac{(\alpha - \mu_1\gamma_1)y + \beta V + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1} \equiv \frac{\alpha y + (\beta + \mu_1^2\gamma_1)V}{\alpha + \beta + \mu_1^2\gamma_1} - \frac{\mu_1\gamma_1 S_1}{\alpha + \beta + \mu_1^2\gamma_1}$$

In Round 2 a trader observes p_2, which, as seen in Eq. 8.1c, is a function of p_1. Given p_1 and the fact that $S_2 \sim N(0, \frac{1}{\gamma_2})$, he infers from $p_2 = \kappa_2(\lambda_2 y + \mu_2 V - S_2 + \xi_{21}p_1)$ that

$$[1/(\kappa_2\mu_2)](p_2 - \kappa_2\lambda_2 y - \kappa_2\xi_{21}p_1) = V - [S_2/\mu_2] \sim N(V, 1/(\mu_2^2\gamma_2))$$

He now updates Eqs. 8.4a and 8.4b. Since supply shocks are IID, the updated posterior is standard

$$E_2^i(V|H_2^i) = \frac{\left[\frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta x^i + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1}\right](\alpha + \beta + \mu_1^2\gamma_1) + \frac{1}{\kappa_2\mu_2}(P_2 - \kappa_2\lambda_2 y - \kappa_2\xi_{21}p_1)(\mu_2^2\gamma_2)}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}$$

Simplification leads to

$$E_2^i(V|H_2^i) = \frac{[\alpha - \mu_1\gamma_1\lambda_1 - \mu_2\gamma_2\lambda_2]y + \beta x^i + [\frac{\mu_1\lambda_1}{\kappa_1}p_1 + \frac{\mu_2\gamma_2}{\kappa_2}p_2 - \mu_2\gamma_2\xi_{21}p_1]}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2} \tag{8.5a}$$

$$\mathrm{Var}(V|H_2^i) = \frac{1}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2} \tag{8.5b}$$

Finally, to compute Eq. 8.1c we average Eq. 8.5a to conclude that

$$\overline{E}_2(V) = \frac{[\alpha - \mu_1\gamma_1\lambda_1 + \mu_2\gamma_2\lambda_2]y + \beta V + [\frac{\mu_1\lambda_1}{\kappa_1}p_1 + \frac{\mu_2\lambda_2}{\kappa_2}p_2 - \mu_2\gamma_2\xi_{21}p_1]}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2} \tag{8.6a}$$

$$\overline{E}_1(p_2) = \kappa_2(\lambda_2 y + \mu_2 \overline{E}_1(V) + \xi_{21} p_1) \qquad \textbf{(8.6b)}$$

We now solve for prices by inserting Eqs. 8.6a and 8.6b into Eq.8.3. The final step is to match coefficients of the price functions and identify $(\kappa_1, \lambda_1, \mu_1, \kappa_2, \lambda_2, \mu_2, \xi_{21})$. For more details of these computations see Allen et al. (2006), Appendix A. This verifies that prices are indeed normally distributed as in Eqs. 8.1a and 8.1b.

What is the length of memory in prices? Multiple trading rounds provide opportunities to deduce more information from prices about V. As trading continues, information on *all past prices is used,* since prices depend on all past unobserved supply shocks. In such a case, the price system is not a finite memory Markov process. The model has been extended to multiperiod trading whereby V is revealed N periods later (see Brown and Jennings, 1989; Grundy and McNichols, 1989; He and Wang, 1995; and Allen et al., 2006). In these models the complexity of inference depends on the presence of a hedging demand of long-lived traders. However, for both long- and short-lived traders, *the number of trading rounds is an arbitrary modeling construct.* It would thus be instructive to examine the limit behavior. In a third round of trading by short-lived traders, the price map becomes

$$p_3 = \kappa_3(\lambda_3 y + \mu_3 V - S_3 + \xi_{31} p_1 + \xi_{32} p_2)$$

Hence, the independent supply shock leads to an updating rule, which is again standard:

$$E_3^i(V|H_3^i) = \frac{E^i(V|H_2^i)(\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2) + \frac{1}{\kappa_3\mu_3}(P_3 - \kappa_3\lambda_3 y - \kappa_3\xi_{31}p_1 - \kappa_3\xi_{32}P_2)(\mu_2^3\gamma_3)}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2 + \mu_3^2\gamma_3}$$

By expressing individual and market forecasts in terms of the unobserved variables, one can easily extend the above to N rounds of trade, and it can be shown that they take the general form

$$E_N^i(V|H_N^i) = \frac{\alpha y + \beta x^i + \sum_{j=1}^{N} \mu_j^2 \gamma_j V}{\alpha + \beta + \sum_{j=1}^{N} \mu_j^2 \gamma_j} - \frac{\sum_{j=1}^{N} \mu_j \gamma_j S_j}{\alpha + \beta + \sum_{j=1}^{N} \mu_j^2 \gamma_j} \qquad \textbf{(8.7)}$$

A standard argument shows the μ_j converge. For simplicity assume $\gamma_j = \gamma$. The independence of the noise S_j together with Eq. 8.7 and the law of large numbers imply that with probability 1 the first term converges to V and the second converges to 0. Hence, in the limit, with probability 1, all forecasts converge to the true V and the effect of the public signal y disappears. This proves that *repeated trading leads to a full revelation of the true value* and that in the limit, $p = V$ and at that time traders do not forecast prices at all. With repeated trading the effect of y disappears. If the unit of time is short, like a month, trading rounds are not really limited. Hence this result contradicts the claim (e.g., see Allen et al., 2006) that the effect of the public signal y on prices lingers on forever.

Allen et al. (2006) use the model to explain the Keynes (1936) Beauty Contest. To see how, recall that $\overline{E}(S_i) = 0$. Then Eq. 8.3 implies that if there are N rounds of trade, then

$$p_1 = \overline{E}_1\overline{E}_2 \ldots \overline{E}_N(V) - [\mathrm{Var}_1(p_2)/(\tau)]S_1 \tag{8.8}$$

The authors then propose that Eq. 8.8 represents the Beauty Contest metaphor, since the price is not equal to the market expectations of V but rather to the average expectation of what the market expects the average expected value of V will be in the future. We comment on this interpretation in Section 8.3.5.

What have we learned so far? The key conclusions of the private information paradigm are that equilibrium prices vary with each date's true intrinsic value of the asset and with the random supply shock of that date, both of which are fundamental factors. Private information, as such, has no separate direct effect on price volatility since the effect of private information is averaged out by the law of large numbers. Hence, the model does not possess the endogenous amplification we seek. In addition, since supply shocks are never observed, the repeated inference causes prices to have infinite memory. In applications such as Brown and Jennings (1989) and Grundy and McNichols (1989), this property was used to explain the phenomenon of "Technical Analysis" defined as the traders' use of past prices, in addition to today's price, to form their demand. More broadly, the static model of asymmetric private information was used in widely diverse applications. One of the more celebrated application in macroeconomics are the Phelps (1970) and Lucas (1972) island models.

8.2.2. Dynamic Infinite Horizon Models

In the model of Section 8.2.1, trades can occur, but it is not a truly dynamic model. Extensions to infinite horizon were developed for many applications, and to get a sense of the issues involved, we review two very different applications. We start with Wang's (1994) study of trade volume.

Wang's (1994) aim is to overcome the no-trade theorems of REE and explain the volume of trade in asset markets. With REE perspective, his hypothesis is that trade is the result of asymmetric private information. Using his notation, he assumes that agent i maximizes expected utility over consumption flows $-E_t^i[\sum_{s=0}^{\infty} \beta^s e^{-\gamma c_{t+s}^i} | H_t^i]$ where expectations are conditioned on information of i. Wang (1994) assumes that there are two assets with payoff in consumption units. A riskless asset with infinitely elastic supply pays a constant rate r and where $R = 1 + r$. The second asset is a risky stock with a fixed supply set at 1, which pays a dividend D_t at date t. The law of motion of dividend is

$$D_t = F_t + \varepsilon_{D,t} \quad \text{where} \quad F_t = a_F F_{t-1} + \varepsilon_{F,t}$$

$(\varepsilon_{D,t}, \varepsilon_{F,t})$ are IID normally distributed, zero mean shocks. Here F_t is the persistent component of the dividend process and $\varepsilon_{D,t}$ is the transitory component. The structure of information is intended to ensure that a closed-form solution is possible. To that end Wang (1994) assumes that there are two types of investors. I-investors have perfect private information and observe F_t. The U-investors receive only a noisy signal about F_t in the form $S_t = F_t + \varepsilon_{S,t}$ where $\varepsilon_{S,t}$ are IID normal, zero mean shocks. Since all investors observe the dividends, the I-investors observe the persistent as well as the transitory components of dividends, whereas the U-investors observe neither.

In addition to the public asset, Wang (1994) assumes the I-investors have a private production technology that is risky and constant returns to scale. If they invest at t the amount I_t, they receive at $t+1$ the amount $I_t(1 + r + q_{t+1})$, where excess return on the private technology q_{t+1} follows:

$$q_{t+1} = \Xi_t + \varepsilon_{q,t+1} \quad \text{and} \quad \Xi_{t+1} = a_\Xi \Xi_t + \varepsilon_{\Xi,t+1}$$

$(\varepsilon_{q,t+1}, \varepsilon_{\Xi,t+1})$ are IID normal, zero mean shocks. Expected excess return Ξ_t is observed only by the I-investors. This sharp information structure is called *hierarchical* since it requires one class of investors to permanently have inferior information. The economy's structure is common knowledge, and all agents are Bayesians with normal priors about parameters they do not know.

Two forces are used to explain the volume of trade. First, asymmetric information between the U- and the I-investors is measured by $\sigma_S^2 = \text{var}(\varepsilon_{S,t})$. If $S_t = F_t$, information about the stock is *symmetric* and $\text{var}(\varepsilon_{S,t}) = 0$. When $\sigma_S^2 > 0$ we have $S_t \neq F_t$ and information is asymmetric. Second, the private technology of the I-investors is unavailable to the U-investors. The effect of this factor on asset demands operates via the correlation between private excess returns q_{t+1} and dividends, measured by $\sigma_{D,q}$—the covariance of $\varepsilon_{D,t+1}$ with $\varepsilon_{q,t+1}$. To understand how this correlation impacts asset demands and trade, suppose that $\sigma_{D,q} \neq 0$ and Ξ_t increases leading I-investors to increase investments in private technology because expected return on such investments increased. But due to $\text{Cov}[\varepsilon_{D,t+1}, \varepsilon_{q,t+1}] \neq 0$, such increased investment changes the risk posture of their portfolio, calling for control of the risk by changing their investments in the publicly traded stock. If $\sigma_{D,q} > 0$, control of risk leads to lower investments in the stock; if $\sigma_{D,q} < 0$, it leads to increased demand for the stock.

The effect of asymmetric information about the private technology is thus due to the need of the I-investors to control their risk, whereas the U-investors are unable to distinguish between changes in F_t and Ξ_t. It shows that the setup of public and private technologies is crucial for Wang's (1994) results. Without private technology, just the asymmetry $S_t \neq F_t$ does not lead to trade, since in this case the uninformed investors deduce F_t from prices, so Wang's (1994) REE becomes fully revealing and we are back to no trade. With $\sigma_{D,q} \neq 0$ the price is linear in F_t and Ξ_t, and uninformed investors are "confused" and cannot deduce either one from the price. This confusion of the U-investors makes it impossible for them to determine the cause of price changes. The U-investors now use the history of the process to conduct a Kalman Filtering to form expectation of their unobserved F_t. In sum, Wang (1994) shows that the model generates trade due to the exogenous shocks F_t and Ξ_t, which cause time variability in the investment composition of the I-investors.

We pause briefly to examine the *causes* of price and volume volatility in models of noisy rational expectations equilibria. In the earlier models, equilibrium prices, such as

Eqs. 8.1a and 8.1b, vary only with exogenous shocks to supply. In Wang (1994) prices vary only in response to exogenous shocks F_t and Ξ_t. This result continues to hold in all other dynamic models of private information, such as He and Wang (1995). What is the effect of increased *diversity* of private information? In the earlier models, private information was so diverse that the law of large numbers was invoked and, as a result, private information had no effect on prices. If diverse private information is to be the cause of trade, we would expect that *increased* diversity of private information should *increase* the volume of trade. The problem is that Wang (1994, page 145) shows the opposite: *Increased* diversity of private information *decreases* the volume of trade. This results from the fact that rising diversity of private information causes uninformed traders to have rising difficulties in deducing from prices useful information needed for trade. We thus conclude that in noisy REE price volatility and volume of trade are caused by exogenous shocks, whereas *diverse private information does not cause or explain them.* We are thus back to the standard model without any endogenous amplification effect.

A second example is Woodford (2003), who revisits the Lucas (1972) model. It is motivated by the fact that Lucas (1972) explains transitory effects of monetary policy but fails to account for the observed fact that monetary disturbances have persistent real effects. Woodford assumes that agents are Dixit-Stiglitz monopolistic competitive price setters who select the nominal price of their product but cannot set the real price since they do not observe the aggregate price level and aggregate output. Although producers cannot observe the real price, their own output is determined by the real price. Aggregate *nominal* GNP is the exogenous state variable (e.g., determined by monetary policy). In equilibrium date t aggregate price level and aggregate output are functions of date t nominal GNP and of all higher-order market expectations (i.e., market expectations of market expectations of . . .) about it. As in Lucas (1972), agents cannot observe nominal GNP and receive private signals about it. Being rational, they learn from the available information and, as in Wang (1994), use a Kalman filtering procedure to learn about the unobserved state variable. With incomplete learning of the persistent exogenous nominal GNP, Woodford (2003) demonstrates persistent money nonneutrality.

Limited space prevents our discussing other applications. Examples include Townsend (1978, 1983); Amato and Shin (2006); Hellwig (2002, 2005) and Angeletos and Werning (2006), who study business cycles; Morris and Shin (2002, 2005), who study the transparency of monetary policy; Singleton (1987), who studies bond markets; Bacchetta and van Wincoop (2006); Hellwig, Mukherji, and Tsyvinski (2006), who study the volatility of foreign exchange rates; and many other policy-oriented papers using global coordination games. These applications use persistent belief heterogeneity to explain the behavior of market *aggregates*. However, why is it asymmetric private information that should provide a basis for heterogeneity? With asymmetric private information agents clearly make different forecasts. Therefore there is the temptation to assume private information to model diversity, and many authors have done just that. This is so common that for some, agents with different opinions are *synonymous* to agents with different private information. Such identification should be rejected. Private information is a very sharp sword that must be used with care. As we have seen, deducing information from prices is complicated and should be employed only when well justified. The virtual equivalence between belief diversity and private information is particularly wrong in macroeconomic applications when agents are assumed to have asymmetric private information with respect to aggregate variables such as interest rates or growth rate of GNP. We have serious doubts about the applicability of the private information paradigm to study asset market dynamics and will now pause to evaluate the results derived so far.

8.2.3. Is Asymmetric Information a Satisfactory Theory of Market Dynamics?

In questioning the use of asymmetric private information assumption, we recall that phenomena studied with private information include the volatility of asset price indices, interest rates, risk premia, foreign exchange rates, business cycles, and the like. In such models individual agents forecast aggregate variables. We thus break our query into two questions. First, does the model of asymmetric private information deliver a satisfactory mechanism of market volatility? Second, as a modeling device, is it reasonable to assume that economic agents have private information about such aggregate variables?

Starting with the first question, our answer is no. In all noisy REE with diverse and independent private signals, asymmetric private information has no impact either on price volatility or on volume of trade. In general, *increased* diversity of private information *decreases* volatility and volume of trade. In any noisy REE, all dynamic characteristics are fully determined by either the standard exogenous shocks such as dividends or exogenous "noise," which is often questionable if it is unobserved. Since we aim to explain excess volatility of markets with mechanisms of *endogenous amplification*, it follows that the asymmetric private information paradigm does not offer such amplification; rather, it leads us simply back to the traditional causes of market dynamics. We examine the second question by outlining five problems raised by models of noisy rational expectations equilibria:

1. *What is the data that constitute "private" information*? For the case of individual firms, the nature of private information is clear, and we discuss it under 2. Now, if forecasters of GNP growth or future interest rates use private information, one must specify the data to which a forecaster has an exclusive access. Without it one cannot interpret a model's implications, since all empirical implications, are deduced from restrictions imposed by private information. In reality it is difficult to imagine the data that constitute private information.
2. *Without correlation, private information explains little.* Even if some agents have some private information about some firms, an aggregate model may have no implications for market dynamics. To deduce any implications, private information has to be repetitive over time, correlated, and widespread. There is no *empirical* evidence for that. Indeed, all models of noisy REE assume private information to be IID distributed, and in that case private information has no effect on volatility, asset pricing, or any other dynamic characteristics.

3. *Asymmetric information implies a secretive economy*. Forecasters take pride in their models and are eager to make their forecasts public. Consequently, there are vast data files on market forecasts of many variables. In discussing public information, forecasters explain their interpretation of information, which is often framed as "their thesis." In contrast, an equilibrium with private information is secretive. Agents do not divulge their private information, since it would deprive them of the advantage they have. In such an equilibrium, private forecast data are treated as sources of *new information* used by all to update their own forecasts. The fact that forecasters are willing to reveal their forecasts is not compatible with private information being the cause of persistent divergence of forecasts.
4. *If private signals and noise are unobserved, how could common knowledge of the structure be attained, and how can we falsify the theory*? For us to deduce private information from public data, the *structure* of private signals must be common knowledge. For example, in Section 8.2.1, $x^i = V + \varepsilon^i$, where ε^i are pure noise, independent across i. A simple question arises: If these are not publicly observed signals, how does common knowledge come about? How does agent i know her own signal takes the form $x^i = V + \varepsilon^i$ and that the signal of k is $x^k = V + \varepsilon^k$? Also, if the crucial data of a theory are not observable, how can one falsify the theory? What, then, are the true restrictions of the theory?
5. *Why are private signals more informative than audited public statements*? Most results of models with private information are based on the assumption that private signals are more informative than public information. For example, in the model of Section 8.2.1, the public signal is $y = E(V)$, where V is unknown. Knowing y is inferior to knowing V. The private signals are $x^i = V + \varepsilon^i$ with ε^i IID and there is a continuum of traders. It is then assumed that "the market" aggregates the private x^i and learns V; hence equilibrium price is a function of the unknown V. This procedure raises two questions:
   - Why are private signals more precise than the professionally audited public statements?
   - Who does the aggregation and knows the IID structure needed for aggregation? If that agent is a neutral agent, why does he not announce V? Or if he is not neutral, he should be part of the model.

In summary, models with an asymmetric private information paradigm fail to explain the observed volatility and the assumptions made have questionable empirical basis. Hence, we must conclude that asymmetric private information is not a persuasive assumption for modeling market dynamics.

8.3. DIVERSE BELIEFS WITH COMMON INFORMATION: THE GENERAL THEORY

Rational expectations and behavioral economics have staked out two extreme positions in contemporary thinking. Under *rational expectations* people know all structural details needed for perfect forecasting, whereas under *behavioral economics* they are driven by psychological impulses that lead to irrational behavior. The theory of *rational belief* offers an intermediate concept of rationality that begins with imperfection of human knowledge, assumes people optimize given the limited knowledge they have, and concludes with a recognition that with imperfect knowledge *rational people make mistakes*. Rational mistakes may be magnified to a point where changing perceptions dominate public life and asset markets. This is the road to rational diverse beliefs and endogenous amplification that we explore. Before proceeding, we mention the papers of Harrison and Kreps (1978), Varian (1985, 1989), Harris and Raviv (1993), Detemple and Murthy (1994), Kandel and Pearson (1995), and Cabrales and Hoshi (1996). These, together with the early writers mentioned in the Introduction, recognized the importance of diverse expectations. We do not review them since they did not anchor the theory with a concept of rationality. In this connection we also note the *adaptive equilibrium* model of Brock and Hommes (1997, 1998) in which agents are boundedly rational.

8.3.1. A Basic Principle: Rational Diversity Implies Volatility

In contrast with asymmetric private information models, we now explain that theories of rational diverse beliefs provide a mechanism for endogenous amplification of volatility. Start by noting that any model discussed here assumes that agents do not know a true probability and hold diverse beliefs about it. This induces two basic questions. First, why do agents not know what they do not know? Second, what is their common knowledge basis? Before proceeding to these questions we clarify our notation. The symbols Π and m are reserved for special probability measures over infinite sequences to be defined shortly. Letters such as Q or P describe probabilities over *infinite sequences*. However, it is useful to think of a "belief" as a collection of conditional probabilities. Hence, instead of i's belief Q^i, we also use terms like *belief* or *date t belief* in reference to date t conditional probability $Q^i(\cdot|H_t)$, where H_t is date t information. Here *belief*, or *date t belief*, refers to a density, a joint distribution or transition function at t that is a component of Q^i. This abuse of notation avoids multiple definitions of belief and should not be confusing when the context is clear. We now return to the two questions.

Starting with the second question, although assumptions about what is common knowledge vary, one answer is general: It is *past data on observed variables*. There is a vector of observable economic variables $x_t \in \mathbb{R}^N$ over time with a data-generating process under a true unknown probability Π on infinite sequences. At t, agents have a long history of past observations $(x_0, x_1, \ldots, x_t)$, allowing rich statistical analysis. Given this data, all agents compute the same finite dimensional distributions of the data; hence all know the same empirical moments, *if they exist*. They then deduce from the data an empirical probability on infinite sequences denoted by m, which is then the empirical common knowledge of all. It will be seen later that m is stationary. Turning now to the first question, the basic cause of diverse beliefs is the fact that m and Π *are not the same*. We briefly explain why.

Our economy has undergone changes in technology and institutions, and these have had deep economic effects, rendering the data process $\{x_t, t = 1, 2, \ldots\}$ nonstationary. Although this means that the distributions of the x_ts are time dependent[1], a simple way to express it is to say that the data process constitutes a sequence of "regimes." But each "regime" is relatively short, with insufficient data to enable agents to learn each of these regimes with any degree of precision. Just to recall a sample of environments we have witnessed in recent years, note that before 1973 to 1979 we had never seen oil shocks and before the 1980s we had never encountered a S&L crisis of the size we had. The dot-com technology cycle of 1996 to 2001 resulted from the novelty of the Internet and the market's failure to predict the timing of its effect: Google was not even a factor then. Finally, the current subprime mortgage crisis results from the fact that the securitization it generated has never been seen before. One source of the crisis is the fact that there is no prior data with which we can predict with accuracy the effect of lower home prices on the rate of default of these securities. In short, it is impossible to learn the unknown probability Π. The stationary probability m (if it exists) is then just an average over an infinite sequence of changing regimes. It reflects long-term frequencies, but it is not the true probability under which the data are generated. Belief diversity arises when agents believe that m is not the truth and the past is not adequate to forecast the future. All surveys of forecasters show that subjective judgment contributes more than 50% to final forecasts (e.g., Batchelor and Dua, 1991). Individual subjective models are thus the way agents express their interpretation of the data. Being common knowledge, the stationary probability m is a reference point for any rationality concepts.

Is it rational to believe that m is the truth? Those who believe that the economy is stationary hold this belief. The theory of *rational beliefs* (see Kurz, 1994, 1997a) defines an agent to be *rational* if her model cannot be falsified by the data m. The theory then has a simple implication that addresses the crucial question of dynamics. It says that an agent's date t belief *cannot be constant or time invariant* unless she believes the stationary probability m is the truth. To see why, consider an agent who holds a constant belief at date t (e.g., time-invariant transition function), which is different from the one implied by m. Since it is constant, the time average of his belief is not m. Since m is the time average in the data, this *proves* that the agent is irrational. In simple terms, it is irrational to be permanently optimistic or pessimistic relative to m. By implication, if a rational agent's belief persists in disagreeing with m, then such a belief *must fluctuate over time around* m. Hence rationality induces dynamic fluctuations on the level of individuals! Now assume that a population exhibits a persistent diversity of beliefs across agents. It implies that most hold beliefs that disagree with m. But then, we have seen that this requires individual beliefs to fluctuate over time. Finally, for an aggregate effect of beliefs, we only add the empirically established fact (see Section 8.3.4, "Individual States of Belief") that individual beliefs are correlated, and this leads to the conclusion that *rational diversity implies aggregate dynamics.*

[1]The technical definition of "nonstationary" that we use requires the process to be time dependent, and this is the customary terminology in ergodic theory and stochastic processes. It is different from the use of this term in the time series literature, which requires the process to have infinite variance.

Diversity of beliefs without private information is often questioned by asking how agents could be wrong and rational at the same time. The idea that rational agents may be wrong relative to an unknown truth is a central component of the theory. Indeed, when rational agents hold diverse beliefs while there is only one true probability law of motion, *then most agents are wrong most of the time*. Since agents' beliefs are correlated, the average market belief is also often wrong, and this is the source of endogenous propagation of market risk and volatility, called *endogenous uncertainty* by Kurz (1974) and Kurz and Wu (1996). Sections 8.3.2 and 8.3.3 provide a precise outline of these ideas.

8.3.2. Stability and Rationality in a General Nonstationary Economy

In a stationary economy, joint probabilities are time invariant and the *Ergodic Theorem* holds: Time averages equal expected values under the true probability, and this probability is deduced from relative frequencies of events. In such environments, the empirical distribution reveals the truth, and since human history is long, agents learn the structure from the data. The fact is that the real data-generating mechanism is not stationary, and history, matters. The question is then how can we discuss rationality and empirical distributions in a complex environment? What is the regularity we can use for analytical evaluation? The answer leads to a definition of *rational beliefs*, which we outline now. The development in this section is based on the material in Kurz (1994).

Let $x_t \varepsilon X \subseteq \mathbb{R}^N$ be a vector of the N observables and let $x = (x_0, x_1, x_2, \ldots)$. Let a future sequence from t on be $x^t = (x_t, x_{t+1}, x_{t+2}, \ldots)$, hence $x^0 \equiv x$. The history to date t is defined by $(x_0, x_1, x_2, \ldots, x_t)$. Let X^∞ be the space of infinite sequences x, and let $\mathfrak{B}(X^\infty)$ be the Borel σ-field of X^∞. Events in $\mathfrak{B}(X^\infty)$ are denoted by letters U, S, T, and so on. For an event $S \varepsilon \mathfrak{B}(X^\infty)$ define the sets $S^{(k)} = \{x | x^k \varepsilon S, k \geq 0\} \equiv$ the event S occurring k periods later. Clearly, $S = S^{(0)}$.

Definition 8.1. A set $S \varepsilon \mathfrak{B}(X^\infty)$ is said to be invariant if $S^{(1)} = S$.

Definition 8.2. A probability Π on $(X^\infty, \mathfrak{B}(X^\infty))$ is said to be *ergodic* if for any invariant set S we have $\Pi(S) = 1$ or $\Pi(S) = 0$.

Throughout the discussion we assume ergodicity so as to simplify the exposition. Under this assumption we develop the basic equivalence theorem, which is the basis of the theory of rational belief. We start with the concept of *statistical stability*. For any finite dimensional set $U \varepsilon \mathfrak{B}(X^\infty)$ define the following time average:

$$m_n(U)(x) = \frac{1}{n} \sum_{k=0}^{n-1} 1_U(x^k) = \left\{ \begin{array}{l} \text{The relative frequency that } U \text{ occurred} \\ \text{among } n \text{ observations since date 0} \end{array} \right\}$$

where

$$1_U(y) = \begin{cases} 1 \text{ if } y \in U \\ 0 \text{ if } y \in U. \end{cases}$$

Although the set U is finite dimensional, it can be a complicated set. For example:

$$U = \left\{ \begin{array}{l} \text{State 1 today} \leq \$1, \text{State 6 next year} \geq 16, \\ 2 \leq \text{State 14 five years later} \leq 5 \end{array} \right\}$$

Definition 8.3. (Property 1) A probability Q on $(X^\infty, \mathfrak{B}(X^\infty))$ is statistically stable if for each cylinder (i.e., finite dimensional) set $U \in \mathfrak{B}(X^\infty)$:

$$(1) \lim m_n(U)(x) = m^Q(U)(x) \text{ exists} \qquad Q \text{ a.e.}$$

Since by ergodicity $m^Q(U)(x)$ is Q a.e. independent of x, we add the notation:

$$(2)\ m^Q(U)(x) = m^Q(U) \qquad Q \text{ a.e.}$$

The restriction to finite-dimensional sets results from the fact that we have only finite data; hence we cannot ascertain whether an infinite dimensional event actually occurred.[2] The first property of *statistical stability* means that the process satisfies the conclusions of the ergodic theorem, although it does not satisfy the standard condition of stationarity used to prove it. Hence, data generated by a stable process have the property that relative frequencies of events converge and all finite moments exist.

The limits in Definition 8.3 might not exist for infinite dimensional sets. Hence the set function defined by the preceding limits is not a probability. However, standard extension theorems permit extension of m^Q to a probability measure on $(X^\infty, \mathfrak{B}(X^\infty))$. To avoid multiple notation, we do not distinguish between these two set functions and denote the extension m^Q as well. We also have:

Theorem 8.1. *m^Q is unique and stationary. It is thus called the stationary measure of Q.*

We now introduce the concept of *weak asymptotic mean stationary* probability measure.

Definition 8.4. (Property 2) A probability Q on $(X^\infty, \mathfrak{B}(X^\infty))$ is *weak asymptotic mean stationary* if for each cylinder set $U \in \mathfrak{B}(X^\infty)$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} Q(U^{(k)}) = m_Q(U)$$

exists.

[2]This in contrast to the theory of checking rules (e.g., Dawid, 1985), which can be effectively implemented only "at infinity" after one has infinite number of observations.

Averaging probabilities of future events $U^{(k)}$ is like averaging over date t beliefs over time. Such average is required to converge. The set function $m_Q(S)$ is not a probability, but by extension one deduces a probability on $(X^\infty, \mathfrak{B}(X^\infty))$ that is unique and stationary. Again, use the same notation for the extension. Thus, if Q *is weak and asymptotic mean stationary*, then m_Q is the probability on $(X^\infty, \mathfrak{B}(X^\infty))$ induced by Property 2. A key result of the theory can now be stated:

Theorem 8.2. *Properties 1 and 2 are equivalent and* $m^Q(S) = m_Q(S)$ *for all events* $S \in \mathfrak{B}(X^\infty)$.

Theorem 8.2 says a data-generating process that is statistically stable with probability Q has an associated stationary probability $m^Q = m_Q$, *which can be computed in two different ways*. First, it can be deduced empirically from relative frequencies computed from the data generated by the process. Second, it can be deduced *analytically* by averaging probabilities of each event over future dates. How do we use this equivalence to define the rationality of belief?

The data process under Π is nonstationary and we now assume it is *statistically stable and ergodic*. This is a reasonable assumption because in our economy additional data increase accuracy of statistical analysis and moments exist. Agents do not know Π and compute empirical frequencies from the data. Using extension, they discover from the data[3] the probability m induced by the dynamics under Π. We reserve the notation $m(U)$ for the limit of the empirical frequencies under Π; hence, under our convention $m \equiv m^\Pi$. This is the stationary measure of Π and we refer to it as *the* stationary measure, or the empirical distribution. Although agents have only finite data, *we assume they actually know the limits* $m(U)$ in *Definition* 8.3, an assumption made for simplicity.[4] All have the same data; hence there is no disagreement among agents about the probability m.

If the economy was actually stationary, $m = \Pi$ *but agents could not know this fact.* The fact is that $m \neq \Pi$ and we seek a concept of rationality of belief that requires an agent to hold a belief that is not contradicted by the empirical evidence represented by the probability m.

[3]We always have only finite data that enable agents to compute at date t only $m_t(U)(x)$. This depends on the data used and with time $m_t(U)(x)$ converges to the limit probability. The assumption made in the text is that the data sequence is very long and the probability m is simple enough (i.e., Markov with short memory) that with finite data agents can obtain a good approximation for the limit measure m. The assumption that agents know the limits is very strong and should not be interpreted to mean we assume that agents have an infinite sequence of observations, since in that case agents will consider not only limits on sequences but also limits on all infinitely many possible subsequences. With finite data we can observe only a finite number of subsequences at all dates; for this reason we do not incorporate restrictions that would be implied by limits on subsequences. We also note that in the nonergodic case the data requirement is greater, since then we need data for many alternative sequences x with different starting points, but the basic theory remains unchanged. For details see Kurz (1994).

[4]The assumption that the limit in Definition 8.3 is known to the agents is made to avoid the complexity of an approximation theory. Without this assumption the diversity of beliefs would be increased due to the diverse opinions about the finite approximations that would be made by different agents. In this context we also mention that the assumption of *ergodicity* is also not needed and is not made in Kurz [1994].

Definition 8.5. A probability belief Q is said to be *a rational belief relative to m* if

1. Q is a *weak asymptotic mean stationary* probability on $(X^\infty, \mathfrak{B}(X^\infty))$,
2. $m_Q(S) = m(S)$ for all events $S \in \mathfrak{B}(X^\infty)$.

A *rational* agent holds a belief Q that is compatible with the empirical evidence m if:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} Q(S^{(k)}) = m_Q(S) = m(S)$$

for all cylinder sets $S \in \mathcal{B}(X^\infty)$.

The first of these conditions is checked analytically by averaging date t beliefs and these should average to a probability m_Q. The second condition requires that $m_Q = m$ where m is common knowledge in the market.

Three implication of Definition 8.5 are notable:

1. Π is a *rational belief* and hence rational expectations are also rational beliefs.
2. m is a *rational belief*, although it is not the truth, since $m \neq \Pi$.
3. A rational belief is generally not equal to Π, showing that agents are wrong and rational at the same time. Agents holding *rational beliefs* disagree more about the short-term forecasts of variables than about forecasts of long-run averages of these same variables. For example, we expect greater disagreement about the forecast of one-year inflation or output growth than about averages of these variables over the next ten years.

Applications of these concepts requires a simplification of the general rationality conditions in Definition 8.5. We thus start with two examples that use a method due to Nielsen (1996).

EXAMPLE 8.1

Agents observe a black box generating numbers x_t in $\{0,1\}$ without long-term serial correlation between x_t and x_{t+k} all $k > 0$. Using a long dataset, they find the mean is 0.5. The probability m is then the probability measure induced by a sequence of IID random variables on $\{0,1\}$ with probability of 1 being 0.5. If the box contains a single coin, the x sequence has an empirical distribution of an IID fair coin, which is the truth. What other processes generate the same empirical measure? As an example, consider a belief in a two-coin family that uses *the realization* of an IID sequence of random variables $g_t, t = 0, 1, \ldots$ in $\{1,2\}$ with probability of 1 being, say, 1/3. A sequence is produced in advance, and is therefore known to the agent. Pick an infinite sequence $g = (g_0, g_1, g_2, \ldots)$. These realizations of $g_t = 1$ or $g_t = 2$ are treated as *fixed parameters* of a new process. It is defined by a process $\{v_t \in \{0,1\}, t = 0, 1, \ldots\}$ with two IID coins in the box that appear at different times, depending on the $g = (g_0, g_1, \ldots)$. v_t is then a sequence of independent random variables of the form

$$P\{v_t = 1\} = \begin{cases} 0.60 & \text{if} \quad g_t = 1 \text{ (coin type 1)} \\ 0.45 & \text{if} \quad g_t = 2 \text{ (coin type 2)} \end{cases} \tag{8.9a}$$

Since $(1/3)(0.60) + (2/3)(0.45) = 0.50$, the empirical distribution is the same as m for almost all g and v. It is easy to see that instead of two possible "regimes" we could have an infinite number of regimes. Note that we have not even specified the true probability Π.

EXAMPLE 8.2

The data x reveal that the empirical probability m is represented by, say, a 10-dimensional matrix M. Again, select an IID sequence of random variables $g_t, t = 0, 1, \ldots$ in $G = \{1, 2\}$ with a probability of 1 being, say, α. Next, construct a joint probability on infinite sequences (g, x) on the space $((X \times G)^\infty, \mathcal{B}((X \times G)^\infty))$, assuming the joint (g, x) process is a stationary Markov process on a 2×10 transition matrix. Suppose that over these 20 (g, x) states the matrix takes the form

$$F = \begin{bmatrix} \alpha F_1 & (1-\alpha)F_1 \\ \alpha F_2 & (1-\alpha)F_2 \end{bmatrix} \tag{8.9b}$$

States in the upper part of F are of $g = 1$ and in the lower part of $g = 2$; hence, the marginal of F on g is the IID distribution $(\alpha, (1-\alpha))$. Now, when $g_t = 1$ the probabilities assigned to x_{t+1} are given by F_1, and when $g_t = 2$ the probabilities of x_{t+1} are given by F_2. The nonstationary probability we seek is represented by the *conditional probability* of F given g. What is the empirical distribution under this conditional probability? Assuming Theorem 8.2 applies we compute the mean probability. With probability α the matrix F_1 is used, and with probability $(1-\alpha)$ the matrix F_2 is used. Hence the stationary distribution implied by the conditional probability of F on g is the expected value $\alpha F_1 + (1-\alpha)F_2$. It follows from Definition 8.5 that F is a *rational belief* if $M = \alpha F_1 + (1-\alpha)F_2$.

Note that in Example 8.1 we define a *rational belief* by knowing in advance the infinite sequence $g = (g_0, g_1, \ldots)$. In Example 8.2 we select only the matrix F so that at date t our forecaster knows his type g_t but is uncertain about g_{t+1}. This approach is the one we follow in the rest of this chapter.

8.3.3. Belief Rationality and the Conditional Stability Theorem

By its own nature, nonstationarity is difficult to describe since it entails a potential infinite variability. Examples 8.1 and 8.2 reveal a simple method to describe nonstationary probability of a real system or as an agent's belief. The question is, how general are such systems, and is there a general principle that generalizes Examples 8.1 and 8.2 to stable but nonstationary systems? The *conditional stability theorem* (Kurz and Schneider, 1996) gives the answer. We explain it now.

Although the theorem holds for general dynamical systems we avoid excessive formalism and discuss here only stochastic processes with probability measures over infinite sequences under which the data are generated.[5] We characterize a family

[5] In the language of Ergodic Theory, the theorem applies to general dynamical systems, but we confine our attention only to dynamical systems under a shift transformation. Since we approach the problem from the point of view of stochastic processes, we avoid the notation of Ergodic Theory altogether.

of probabilities described by sequences of parameters in $G \subseteq \mathbb{R}^L$. These parameters represent structural change and could be thought of as a sequence of "regimes" over time. To do that, consider *joint* sequences of $(x_t, g_t) \in X \times G$ generated under a probability P over the space $((X \times G)^\infty, \mathcal{B}((X \times G)^\infty))$. The following theorem is stated under the assumption that P is stationary, although it is sufficient that it be stable. Now, let P_g be a regular *conditional probability* of x given g. That is:

$$P_g(S) : G^\infty \times \mathcal{B}(X^\infty) \to [0, 1] \quad g \in G^\infty \quad S \in \mathcal{B}(X^\infty) \tag{8.9}$$

such that for each $S \in \mathcal{B}(X^\infty)$, $P_g(S)$ is a measurable function of g and for each g, $P_g(\bullet)$ is a probability on $(X^\infty, \mathcal{B}(X^\infty))$. We now consider the data $(x_t, t = 0, 1, 2, \ldots)$ as being generated under the conditional probability P_g parametrized by g, where we consider g_t as the parameters of the regime in place at date t. The question we ask is, under what conditions is the data-generating system under the probability $P_g(\bullet)$ stable for almost all parameter sequences g?

Before proceeding, we pause and ask how we should think of the joint system. The joint process on data and parameters could be considered in two ways. One is as a description of the way our world evolves, inducing statistical regularity of the data-generating process. The joint system is then a true unobserved law of motion of our economy, the $g_t \in G$ are *unobserved parameters*, and the statistical properties of the parameters are interrelated with the statistical properties of the data. Both arise from the stability of the joint system. Or else, which is the way we use it here, the joint process is a *model* that a rational agent uses to formulate his belief. The parameter g_t then pins down the state of belief or the agent "type" at date t.

To proceed, we need two technical definitions. Let P_X be the *marginal measure* of P on $(X^\infty, \mathcal{B}(X^\infty))$ and P_G be the marginal of P on $(G^\infty, \mathcal{B}(G^\infty))$ defined by

$$\begin{aligned} P_X(S) &= P(S \times G^\infty) \quad \text{for all} \quad S \in \mathcal{B}(X^\infty) \\ P_G(Y) &= P(X^\infty \times Y) \quad \text{for all} \quad Y \in \mathcal{B}(G^\infty) \end{aligned}$$

Our perspective is then simple. The joint is a process on data and parameters under P, but the data $(x_t, t = 0, 1, 2, \ldots)$ is generated under the conditional probability P_g parametrized by g.

Theorem 8.3. *Conditional Stability Theorem (Kurz and Schneider 1996)—Suppose G is countable and the probability P on* $((X \times G)^\infty, \mathcal{B}((X \times G)^\infty))$ *is stationary and ergodic. Then:*

- *The conditional probability P_g is stable and ergodic for P almost all g. The stationary measure of P_g is denoted by m^{P_g}.*
- *m^{P_g} is independent of g P almost all g.*
- $m^{P_g} = P_X$.

A sufficient condition for stability and ergodicity of P_g is then the stability and ergodicity of P. In addition to the stability of the conditional probability P_g, we also have the result that the stationary measure of P_g is the marginal measure P_X. It is well known that for all $S \in \mathcal{B}(X^\infty)$ we have

$$P(S \times G^\infty) = \int_{G^\infty} P_g(S) P_G(dg)$$

Hence, P_X is computed by averaging the conditional probability P_g over frequencies at which it is used, as is the case in Examples 8.1 and 8.2.

The theorem defines a general family of probabilities that are *rational beliefs relative to a known* m. That is, let the data be generated under a stable, ergodic, but unknown probability Π with a stationary measure m. Now an agent formulates a joint process of (x, g) under probability P, which induces, with parameters g, a belief P_g on data sequences x. The question is then under what condition is P_g a stable and ergodic rational belief? Theorem 8.3 tells us that if the joint P is stationary and ergodic, then P_g is stable and ergodic with a stationary measure satisfying $m^{P_g} = P_X$. It is a *rational belief* if the joint satisfies $P_X = m$. The joint is then a model an agent uses to formulate his belief. Our development that follows is based on this way of constructing beliefs relative to a known empirical probability m. But since we also assume Gaussian processes with a continuum of states, we comment on Theorem 8.3's condition that G is countable. In general, Theorem 8.3 is false for continuum state space without more restrictions. For Gaussian processes the theorem holds and we can give a direct proof. A more general theorem is given by Nielsen (2007) for Harris processes.

Theorem 8.3 offers a tractable way to describe beliefs about general asset structures. To simplify exposition, we concentrate in the rest of this chapter on a simple asset structure. To that end we postulate an exogenous environment in which there is a single risky asset or a single risky portfolio of assets paying an exogenous risky payoff $\{D_t, t = 1, 2, \ldots\}$ with a nonstationary and unknown true probability. We assume that the available long history of the data reveals that the empirical distribution of the D_t's constitute a Markov process with transition

$$D_{t+1} = \mu + \lambda_d(D_t - \mu) + \rho^d_{t+1} \quad \rho^d_{t+1} \sim N(0, \sigma^2_d)^6$$

and unconditional mean μ. Let $d_t = D_t - \mu$; then the process $\{d_t, t = 1, 2, \ldots\}$ is a zero mean, nonstationary with unknown true probability Π and empirical probability m. Hence, $\{d_t, t = 1, 2, \ldots\}$ has an empirical distribution that implies a transition function of the first-order Markov process

$$d_{t+1} = \lambda_d d_t + \rho^d_{t+1} \quad \rho^d_{t+1} \sim N(0, \sigma^2_d) \qquad \textbf{(8.10a)}$$

Since the implied stationary probability is denoted by m, we write $E^m[d_{t+1}|d_t] = \lambda_d d_t$.

[6] As will shortly be explained, in many applications the dividend or payoff D_t grows without bound, does not have a finite mean, and has growth rates that have an empirical distribution characterized by a stationary transition of a Markov process. The same applies to other statistically stable processes with trends, in which case the concept of stability is applied to growth rate data, not to the absolute quantities.

We also review papers that assume finite state spaces. In simulation work which study volatility it is assumed $D_{t+1} = v_{t+1} D_t$, where v_t is the random *growth rate* of dividends, which is a Markov process over a finite state space. In studying the equity risk premium Mehra and Prescott (1985) assume v_t takes two values. They find that the long-term empirical distribution is represented by a stationary and ergodic Markov process over a state space that reflects extreme business cycle states of "recession" and "expansion." They estimate the transition matrix of the two states to be

$$\begin{bmatrix} \varphi & 1-\varphi \\ 1-\varphi & \varphi \end{bmatrix} \quad \varphi = 0.43 \qquad \textbf{(8.10b)}$$

Is the stationary model of Eq. 8.10a or 8.10b the true process? Those who believe the economy is stationary would accept Eq. 8.10a or 8.10b as the truth. Most do not believe past empirical record is adequate to forecast the future, and this leads to nonstationary and diverse beliefs. The problem is, then, how do we describe an equilibrium in such an economy? The belief structure is our next topic.

8.3.4. Describing Individual and Market Beliefs with Markov State Variables

The approach taken by Theorem 8.3 raises a methodological question. In formulating an asset-pricing theory, do we need to describe in detail each agent's model? Are such details needed for a study of price dynamics? Although an intriguing question, we suggest that such details are not needed. To describe an equilibrium, all we need is to specify how beliefs affect agents' perceived stochastic transition of state variables. Once specified, Euler equations are well defined and market clearing leads to equilibrium pricing. Theorem 8.3 leads to this approach by proposing to treat individual beliefs as state variables, *generated within the economy*.[7] This is the approach we now explain.

Start with the fact that agents who hold heterogeneous beliefs are willing to reveal their forecasts when surveyed. We thus assume that distributions of individual forecasts are publicly observable. An individual's belief is described with a *personal* state of belief that uniquely pins down his perception of the transition to next period's state variables. It follows that personal state variables and economy-wide state variables are not the same. A personal state of belief is the same as any other state variables in an agent's decision problem but can also be interpreted as defining the "type" of agent who is uncertain of her future belief type but knows the dynamics of her belief state. The distribution of belief states is then an economy-wide state variable. Endogenous variables depend on the economy's state variables. Hence, moments of the market distribution

[7]In using Theorem 8.3 there are two possible approaches that can be taken. The first is based on Nielsen (1996), who treats the infinite sequence g_t of parameters as fixed and known to each agent in advance, as in Example 8.1. Hence, in Example 8.1 a belief is a Dirichlet distribution in $(G^\infty, \mathcal{B}(G^\infty))$. In this chapter we follow the developments in Kurz and Motolese (2001), Kurz, Jin, and Motolese (2005a, 2005b), and Kurz and Motolese (2007), who treat the sequence g_t as state variables that define the belief of an agent or identify his type. These papers assume that at date t an agent does not know his own type at date $t+1$.

of beliefs may have an effect on endogenous variables such as prices. Also, in a large economy, an agent's *anonymity* means that a personal state of belief is perceived to have a negligible effect on prices and is assumed not to be public. Some papers assume an exponential utility that results in equilibrium endogenous variables depending only on the *mean market belief*. Finally, in the following discussion the set of agents is implicit and not specified: It might be finite or infinite. It is specified when needed.

Individual States of Belief

We introduce agent i's *state of belief* at t, g_t^i, which pins down his transition functions. Apart from "anonymity," we assume that agent ℓ knows his own g_t^ℓ and the *distribution* of the g_t^i over the agents for all past dates $\tau \leq t$. This last assumption is justified by the fact that an infinite horizon economy consists of a sequence of decision makers. An agent knows his states of beliefs but does not know the states of belief of *all* his own specific predecessors. Past belief distributions are public information, since samples of g_t^i are made public. We specify the dynamics of g_t^i by

$$g_{t+1}^i = \lambda_Z g_t^i + \rho_{t+1}^{ig} \quad \rho_{t+1}^{ig} \sim N(0, \sigma_g^2) \tag{8.11}$$

where ρ_t^{ig} are correlated across agents, reflecting correlation of beliefs across individuals. The *state of belief* is a central concept and Eq. 8.11 is taken as a primitive description of type heterogeneity. One can, however, deduce Eq. 8.11 from more elementary principles (see the next subsection).

How does g_t^i pin down the stochastic transition? In various models agent i's *perception* of date t distribution of d_{t+1} (denoted by d_{t+1}^i) is described by using the belief state as follows:

$$d_{t+1}^i = \lambda_d d_t + \lambda_d^g g_t^i + \rho_{t+1}^{id} \quad \rho_{t+1}^{id} \sim N(0, \hat{\sigma}_d^2) \tag{8.12}$$

The assumption that $\hat{\sigma}_d^2$ is the same for all i is made only for simplicity. An agent who believes the empirical distribution is the truth expresses it by $g_t^i = 0$. It follows that given information H_t, the state of belief g_t^i measures the deviation of her forecast from the empirical stationary forecast

$$E^i[d_{t+1}^i | H_t, g_t^i] - E^m[d_{t+1} | H_t] = \lambda_d^g g_t^i \tag{8.13}$$

Eq. 8.13 shows how g_t^i is measured in practice. For any x_t, publicly available data on i's forecasts of x_{t+h}, measure $E^i[x_{t+h}|H_t, g_t^i]$, where h is the forecast horizon. To estimate the difference in Eq. 8.13 one then uses standard techniques such as Stock and Watson (2001, 2002, 2005) to compute the stationary forecast $E^m[x_{t+h}|H_t]$. Average market belief is then computed by averaging the left side of Eq. 8.13 over agents. Fan (2006) and Kurz and Motolese (2007) offer examples of such construction. Figures 8.1, 8.2, and 8.3 illustrate the time series of *average market belief* with horizon of six-month for the six-month Treasury Bill rate, for percent change in the GDP

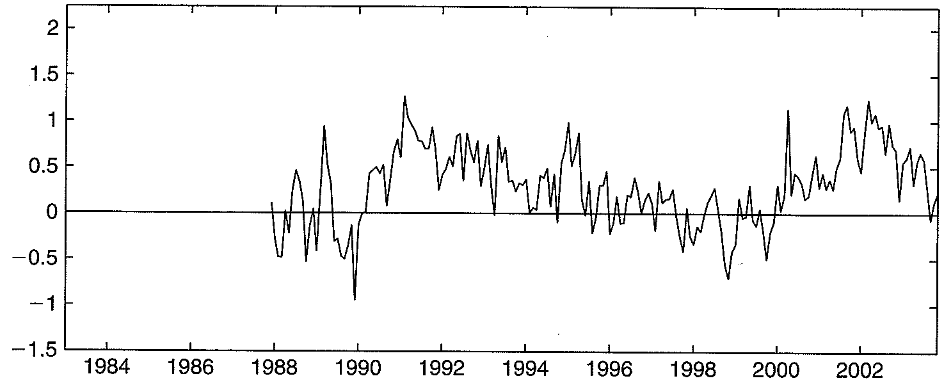

FIGURE 8.1 Six-month Treasury Bill rate: six-months-ahead market belief.

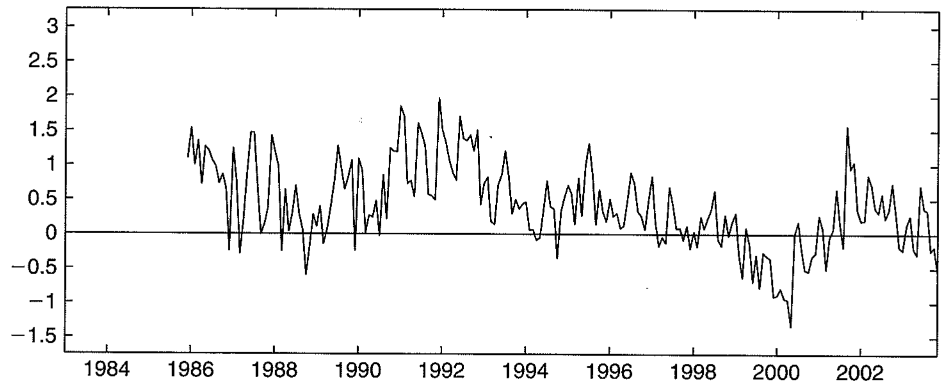

FIGURE 8.2 GDP deflator for inflation rate: two-quarters-ahead market belief.

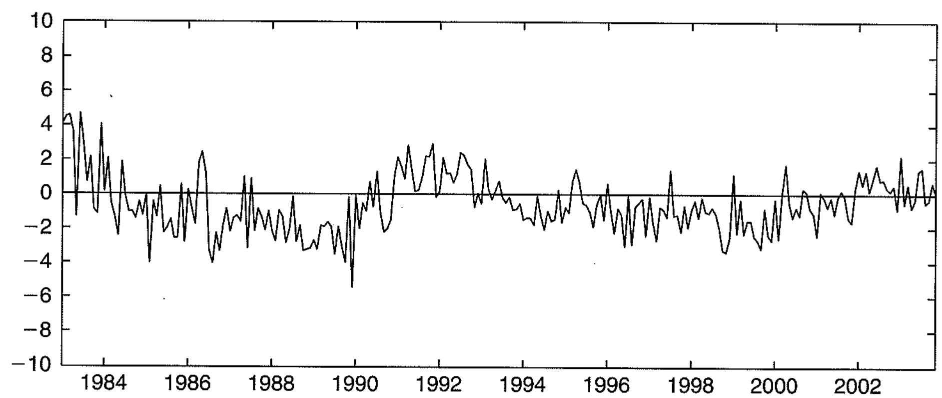

FIGURE 8.3 Month-over-month, annualized growth rate of industrial production: six-months-ahead market belief.

deflator (measuring inflation) and growth rate of industrial production. For each variable, the average forecasts are given by the blue-chip financial forecasts, whereas $E^m[x_{t+h}|H_t]$ is computed by Kurz and Motolese (2007) using methods of Stock and Watson (2001, 2002, 2005).

In Figures 8.1 and, 8.3, average market belief fluctuates around zero as predicted by the theory, *even during the short period at hand.* In Figure 8.2 this pattern is not exactly maintained by market belief about inflation due to persistent deviations of inflation forecasts from the normal pattern during the 1980s and early 1990s. Over a longer horizon, the pattern of fluctuations around zero is restored. All three figures are compatible with the Markov property assumed in Eq. 8.11 (and later in Eq. 8.15).

Note that since belief variables arise from structural change, g_t^i in 1900 has nothing to do with the one in 2000: They reflect different social and technological environments. Also observe that a belief g_t^i is not "information" about unknown structural parameter; rather, it describes the *opinion* of agent i. Hence, agents do not treat individual beliefs of others as *information*, and even if they observed them they do not deduce from them anything about unknown parameters.

Deducing the Dynamics of Individual Belief from Bayesian Inference

Although Eq. 8.11 is a primitive, we can deduce it from elementary principles. There are many ways to do this; we review the approach of Kurz (2007). He shows how subjective interpretation of data arises from public *qualitative* information, which always accompanies the release of quantitative data. To highlight this idea, note first that a Markov property in Eq. 8.11 is not surprising since Bayesian posteriors have a Markov form. This is not sufficient, since if agents knew that the dividend process has an unknown parameter b, which takes the form

$$d_{t+1} - \lambda_d d_t = b + \rho_{t+1}^d \qquad \rho_{t+1}^d \sim N\left(0, \frac{1}{\beta}\right)$$

then a Bayesian posterior would be a *convergent* belief sequence. Hence, the key object is to explain where the *random term* in Eq. 8.11 comes from.

From Eqs. 8.16a and 8.16b, agents know λ_d, and Kurz (2007) assumes they also know that under the true probability Π the transition of d_t is

$$d_{t+1} - \lambda_d d_t = b_t + \rho_{t+1}^d \qquad \rho_{t+1}^d \sim N\left(0, \frac{1}{\beta}\right) \tag{8.14}$$

b_t are the unknown, exogenous time-varying mean values of $d_{t+1} - \lambda_d d_t$ and hence g_t^i are beliefs about b_t. Since there is no universal method to learn a sequence of parameters, Kurz (2007) outlines a Bayesian updating procedure that is supplemented by subjective estimates of $d_{t+1} - \lambda_d d_t$, *which are based on qualitative public information.* We start with the qualitative data.

Qualitative data about all aspects of our economy are provided at all times, and financial markets pay a great deal of attention to them. Profit is just one number in a financial report that covers many additional issues. Firms may announce new research projects, new organizational structures, or new products. Qualitative data are rarely comparable over time. For example, when a firm starts research into a new topic, no past data exist on it. Qualitative information is modeled by Kurz (2007) in the form of qualitative *statements that can potentially impact future profits*. The list of statements may change with time and the impact on profits may be positive or negative. For each statement, a realization at $t + 1$ may be a "success" or "failure" in its effect on profits. Agent i has subjective maps from the list of potential successes or failures to potential future value of $(d_{t+1} - \lambda_d d_t)$. Finally, conditional on the statements, agent i attaches subjective probabilities to *vectors* of successes or failures, and by taking expected value she makes a subjective estimate Ψ_t^i of $(d_{t+1} - \lambda_d d_t)$. Ψ_t^i varies with time since new statements are made each date. Because the long-term average of $(d_{t+1} - \lambda_d d_t)$ is zero, rationality requires the Ψ_t^i to be zero mean random variables. We now examine an alternative Bayesian updating procedure for estimating the same quantity.

Kurz (2007) starts the updating process by assuming that β in Eq. 8.14 is known. At first decision date t (say, $t = 1$), an agent has two pieces of information. He observes d_t and receives public qualitative information with which he assesses Ψ_t^i. Without Ψ_t^i his prior belief at $t = 1$ is normal with mean b. However, to start the process, he uses *both sources* to form a prior belief $E_t^i(b_t|d_t, \Psi_t^i)$ about b_t (used to forecast d_{t+1}). However, the changing parameter b_t leads to a problem. When $d_{t+1} - \lambda_d d_t$ is observed, Agent i updates his belief to $E_{t+1}^i(b_t|d_{t+1}, \Psi_t^i)$[8] in a standard Bayesian procedure. But he needs an *estimate of* b_{t+1}, not of b_t. Hence, his problem is how to go from $E_{t+1}^i(b_t|d_{t+1}, \Psi_t^i)$ to a prior of b_{t+1}. Without new information, his belief about b_{t+1} is unchanged and $E_{t+1}^i(b_t|d_{t+1}, \Psi_t^i)$ would be the new prior of b_{t+1}. This is his first estimate of b_{t+1}. Next the agent observes the qualitative information released publicly before trading at $t + 1$, which provides an alternate subjective estimate Ψ_{t+1}^i of b_{t+1}. Now the agent has two independent sources for belief about b_{t+1}: $E_{t+1}^i(b_t|d_{t+1}, \Psi_t^i)$ and Ψ_{t+1}^i. Kurz (2007) now assumes:

Assumption 8.1. Agent i uses a subjective probability τ to form date $t + 1$ prior belief defined by

$$E_{t+1}^i(b_{t+1}|d_{t+1}, \Psi_{t+1}^i) = (1 - \tau)E_{t+1}^i(b_t|d_{t+1}, \Psi_t^i) + \tau\Psi_{t+1}^i \qquad 0 < \tau \leq 1$$

At $t = 1$ it was assumed that the initial prior mean is b, hence for consistency, if Ψ_t^i is Normal, then

$$b_1 \sim N\left((1 - \tau)b + \tau\Psi_1^i, \frac{1}{\vartheta}\right)$$

for some ϑ.

This assumption is the element that permits $E_{t+1}^i(b_t|d_{t+1}, \Psi_t^i)$ to be upgraded into a prior belief at date $t + 1$, $E_{t+1}^i(b_{t+1}|d_{t+1}, \Psi_{t+1}^i)$, before d_{t+2} is observed. The following result is then shown:

[8] We use the notation $E_t^i(b_t|d_t, \Psi_t^i)$ for the *prior* belief at date t about the unknown parameter b_t used to forecast d_{t+1}. We then use the notation $E_{t+1}^i(b_t|d_{t+1}, \Psi_t^i)$ for the posterior belief about b_t given the observation of d_{t+1}. Assumption 8.1 will then use this posterior belief as a building block in the formation of the new *prior* $E_{t+1}^i(b_{t+1}|d_{t+1}, \Psi_{t+1}^i)$.

Theorem 8.4. (Kurz, 2007): *Suppose* $\Psi_t^i \sim N(0, \frac{1}{v})$, *IID and Assumption (8.1) hold. Then there exists a constant* $0 < \kappa < 1$ *such that for large t the prior belief* $E_t^i(b_t|d_t, \Psi_t^i)$ *is a Markov random variable, and by identifying* $g_t^i = E_t^i(b_t|d_t, \Psi_t^i)$ *and* $(1-\tau)\kappa = \lambda_Z$ *we have that the dynamics of Eq. 8.11 hold: Assumption 8.1 implies Eq. 8.11.*

Theorem 8.4 shows that as the length of the data increases with time, nothing new is learned. The posterior fluctuates forever, but its dynamic *law of motion* converges. That is, in the limit Eq. 8.11 holds and any new data alter the conditional probability of agents but do not change the law of motion of g_t^i. If $\tau = 0$, the agent ignores all qualitative information and the posterior converges. But the results hold no matter how small τ is, since even the slightest perturbation of the Bayesian updating process cause it to fluctuate forever. This result is compatible with the work of Freedman (1963, 1965), who first demonstrated the general nonconvergence of Bayesian posteriors in an IID context but when the parameter space is countable. The work of Acemoglu et al. (2007) also relates to the issue of diversity in a setting in which the data do not permit an identification of the state.

Individual Perceptions, Market Belief, and Endogenous Amplification

We assume that the market is large and anonymous and that the distribution of beliefs is observable; hence its moments are known. Let $Z_t = \int g_t^i di$ be the first moment and refer to it as *average market belief.* Due to correlation across agents, averaging over the agents does not result in a constant, and the average $\epsilon_t = \int \rho_t^{ig} di$ is a random variable, not a constant 0. Hence we have

$$Z_{t+1} = \lambda_Z Z_t + \epsilon_{t+1} \tag{8.15}$$

Correlation of ρ_t^{ig} *across agents* may exhibit nonstationarity, and that would be inherited by the ϵ process. The empirical distribution of the ϵ process is denoted by a process ρ^Z. If the ϵ process is stationary, $\epsilon_t = \rho_t^Z$. Since the Z_t are observable, market participants have data on the *joint* process (d, Z); hence they know their *joint empirical distribution.* We assume that, this distribution is described by the system of equations

$$\begin{aligned} &\textbf{(8.16a)} \quad d_{t+1} = \lambda_d d_t + \rho_{t+1}^d \\ &\textbf{(8.16b)} \quad Z_{t+1} = \lambda_Z Z_t + \rho_{t+1}^Z \end{aligned} \quad \begin{pmatrix} \rho_{t+1}^d \\ \rho_{t+1}^Z \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_Z^2 \end{bmatrix} = \Sigma \right) \quad \text{IID}$$

Eqs. 8.16a–8.16b are the first formal structure to explain the mechanisms of endogenous amplification of volatility. We started with one exogenous shock, and we find that correlation of beliefs expanded the economy's state space to include an aggregate market belief variable. If this variable affects prices, it causes endogenous amplification of market dynamics and volatility. Note that Z_t does not arise from individual choice; rather, it is a market externality arising from the correlation of individual beliefs. Indeed, in the theory reviewed here, the emergence of the distribution of market belief, as an observable variables that has economic impact, is the single most important development. But, to demonstrate that amplification is actually present, we need to show that equilibrium prices depend on market beliefs. This suggests a natural definition that is useful in assessing equilibria:

Definition 8.6. An economy exhibits *endogenous uncertainty* if an equilibrium price map is a function of the market belief.[9]

We now explain agent i's perception model. In Eq. 8.12, g_t^i pins down agent i's forecast of d^i_{t+1}. We now broaden this idea to a perception model of the two state variables (d^i_{t+1}, Z^i_{t+1}) given d_t and Z_t. Following Theorem 8.3, her belief takes the joint form:

$$\begin{aligned} &\textbf{(8.17a)}\quad d^i_{t+1} = \lambda_d d_t + \lambda^g_d g^i_t + \rho^{id}_{t+1} \\ &\textbf{(8.17b)}\quad Z^i_{t+1} = \lambda_Z Z_t + \lambda^g_Z g^i_t + \rho^{iZ}_{t+1} \\ &\textbf{(8.17c)}\quad g^i_{t+1} = \lambda_Z g^i_t + \rho^{ig}_{t+1} \end{aligned} \qquad \begin{pmatrix} \rho^{id}_{t+1} \\ \rho^{iZ}_{t+1} \\ \rho^{ig}_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \hat{\sigma}^2_d & \hat{\sigma}^2_{Zd} & 0 \\ \hat{\sigma}^2_{Zd} & \hat{\sigma}^2_Z & 0 \\ 0 & 0 & \sigma^2_g \end{bmatrix} = \Sigma^i \right)$$

g_t^i defines belief (d_{t+1}, Z_{t+1}), and Eqs. 8.17a and 8.17b show it pins down i's perceived transition of (d^i_{t+1}, Z^i_{t+1}). This simplicity ensures that one state variable pins down agent i's subjective belief; hence

$$E_t^i \begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} - E_t^m \begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda^g_d g_t^i \\ \lambda^g_Z g_t^i \end{pmatrix}$$

We stress two facts. First, market belief is shaped by correlation across individuals, but such correlation is a market externality with implications to efficiency considerations. Second, from the perspective of agents, Z_t is an economy-wide state variable like any other. But market belief is often wrong: It has forecast more recessions than actually occurred. In contrast with asymmetric information models, agents do not use Z_t to update beliefs about future exogenous variables: Eq. 8.17a does not depend on Z_t. Agents do not view Z_t as information about d_{t+1}, since it is not a "signal" about unobserved private information. They do consider Z_t as crucial "news" about what the *market thinks* about d_{t+1}. Since $t+1$ prices depend on $t+1$ market belief, to forecast future endogenous variables an agent must forecast Z_{t+1}, which express future beliefs of other agents.

Rationality Conditions for the Gaussian Model

Theorem 8.3 gives general rationality conditions, and we now explore the specific conditions that must be satisfied by the perception models (Eqs. 8.17a–8.17c). We note first that some rationality conditions have already been imposed. First, we argued that rational agents exhibit fluctuating beliefs, since a *constant* belief that is not the empirical

[9]Earlier we stressed the notion of *endogenous uncertainty* as entailing excess price volatility due to the effect of beliefs. The precise definition as given here was introduced in Kurz and Wu (1996) in the context of a General Equilibrium model. Kurz and Wu (1996) define the term as a property of the price map that has multiple prices for the same exogenous state.

probability is irrational. Second, $g_t{}^i$ are required to have an unconditional zero mean since beliefs are all about deviations from empirical frequencies. Third, any belief is a conditional probability of a stationary joint system. We now turn to Eqs. 8.17a–8.17c.

For Eqs. 8.17a–8.17c to be a *rational belief* it needs to induce the same empirical distribution of the observables (d_t, Z_t) as Eqs. 8.16a and 8.16b. In accord with Theorem 8.3, we then treat $g_t{}^i$ symmetrically with other random variables and require that for Eqs. 8.17a–8.17c to be a rational belief, we must have:

$$\text{Empirical distribution of the process } \begin{Bmatrix} \lambda_d^g g_t{}^i + \rho_{t+1}^{id} \\ \lambda_Z^g g_t{}^i + \rho_{t+1}^{iZ} \end{Bmatrix} = \text{the distribution of } \begin{pmatrix} \rho_{t+1}^d \\ \rho_{t+1}^Z \end{pmatrix} \sim N\left(\begin{matrix}0\\0\end{matrix}, \begin{bmatrix} \sigma_d^2, & 0, \\ 0, & \sigma_Z^2 \end{bmatrix}\right), \text{IID} \tag{8.18}$$

To compute the implied statistics of the model, we first compute the moments of $g_t{}^i$. From Eq. 8.17c, the unconditional variance of $g_t{}^i$ is $\mathbb{V}\text{ar}(g^i) = \sigma_g^2/(1-\lambda_Z^2)$. Hence, we have two sets of rationality conditions that follow from Eq. 8.18. The first arises from equating the covariance matrix

$$\text{(i)}\ \frac{(\lambda_d^g)^2\sigma_g^2}{1-\lambda_Z^2} + \hat{\sigma}_d^2 = \sigma_d^2 \qquad \text{(ii)}\ \frac{(\lambda_Z^g)^2\sigma_g^2}{1-\lambda_Z^2} + \hat{\sigma}_Z^2 = \sigma_Z^2 \qquad \text{(iii)}\ \frac{\lambda_d^g\lambda_Z^g\sigma_g^2}{1-\lambda_Z^2} + \hat{\sigma}_{Zd} = 0$$

The second set arises from equating the serial correlations of the two systems

$$\text{(iv)}\ \frac{(\lambda_d^g)^2\lambda_Z\sigma_g^2}{1-\lambda_Z^2} + \mathbb{C}\text{ov}(\hat{\rho}_t^{id}, \hat{\rho}_{t+1}^{id}) = 0 \qquad \text{(v)}\ \frac{(\lambda_Z^g)^2\lambda_Z\sigma_g^2}{1-\lambda_Z^2} + \mathbb{C}\text{ov}(\hat{\rho}_t^{iZ}, \hat{\rho}_{t+1}^{iZ}) = 0$$

(i) to (iii) fix the covariance matrix in Eqs. 8.17a–8.17c, and (vi)–(v) fix the serial correlation of $(\hat{\rho}_t^{id}, \hat{\sigma}_t^{iZ})$. An inspection of Eqs. 8.17a–8.17c reveals the choice left for an agent are the two parameters $(\lambda_d^g, \lambda_Z^g)$. But under the rational belief theory, these are not free either, since there are natural conditions they must satisfy. First, $\hat{\sigma}_d^2 > 0, \hat{\sigma}_Z^2 > 0$ place two strict conditions on $(\lambda_d^g, \lambda_Z^g)$:

$$|\lambda_d^g| < \frac{\sigma_d}{\sigma_g}\sqrt{1-\lambda_Z^2} \qquad |\lambda_Z^g| < \frac{\sigma_Z}{\sigma_g}\sqrt{1-\lambda_Z^2}$$

Finally, we need to ensure the covariance matrix in Eqs. 8.17a–8.17c is positive definite. The following is a sufficient condition

$$\frac{1-\lambda_Z^2}{\sigma_g^2} > \frac{(\lambda_Z^g)^2}{\sigma_Z^2} + \frac{(\lambda_d^g)^2}{\sigma_d^2}$$

The "free" parameters $(\lambda_d^g, \lambda_Z^g)$ are thus restricted to a narrow range, which is empirically testable.[10]

Comments on the Finite State Space Case

Much of the simulation work reported in Section 8.4 uses finite state space economies.[11] For example, Kurz (1997c), Kurz and Beltratti (1997), Kurz and Schneider (1996), Kurz and Motolese (2001, 2007), Motolese (2003), Nielsen (1996, 2003), Nakata (2007), and Wu and Guo (2003) all use OLG models with two "dynasties" of finite lived agents in which each agent has, at each date, two belief states. They assume that the sequence of parameters $g_t{}^i$ are IID with $Q^i\{g_t{}^i = 1\} = \alpha_i$, but $g_t{}^1$ and $g_t{}^2$ *are correlated.* This marginal distribution is fixed in the following discussion. The empirical distribution of dividends is typically assumed as Markov on two values (d^H, d^L) with a transition matrix as in Eq. 8.10b. We use this matrix together with Example 8.2 to review the main ideas.

Starting with the *endogenous amplification effect,* note that although the exogenous Markov dividend growth rate process takes two values, with two dynasties of agents, each with two belief states, the economy's state space is of Dimension 8. That is, the economy has four market belief states defined by possible values of the pair (g^1, g^2) and eight economy-wide states defined by the identification:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \Leftrightarrow \begin{bmatrix} d = d^H, g^1 = 1, g^2 = 1 \\ d = d^H, g^1 = 1, g^2 = 2 \\ d = d^H, g^1 = 2, g^2 = 1 \\ d = d^H, g^1 = 2, g^2 = 2 \\ d = d^L, g^1 = 1, g^2 = 1 \\ d = d^L, g^1 = 1, g^2 = 2 \\ d = d^L, g^1 = 2, g^2 = 1 \\ d = d^L, g^1 = 2, g^2 = 2 \end{bmatrix}. \quad \textbf{(8.19)}$$

[10]It may appear that the empirical evidence consists of more than the moments of the data series as stipulated in Section 8.3.1 and Definition 8.1; that is, one should look not only at the full data series but also at subsequences. Kurz (1994) argues that economic time series have deterministic patterns in seasonal and cyclical frequencies; hence if these are cleaned out so that we look at seasonally and cyclically and adjusted data, then under ergodicity, with probability 1 the empirical distribution along any subsequence over dates whose selection does not depend on the observed data is the same as the distribution along the entire sequence of data. Also, with finite data there are always an infinite number of unobservable sequences. Hence, there are no new restrictions that can be deduced from looking at subsequences. See Dawid (1985) for the Calibration literature view on the question of rationality conditions along subsequences.

[11]States of belief are described either with finite or continuous state models. Continuous state models tend to be more complex than discrete state models, which are more tractable, but the simulation results of the two models are essentially the same. To avoid repetition we report later detailed results deduced only for the continuous state models. Since a reader may find either one of these two more suitable for his or her application, we describe in the text the basic structure of *both* models. Hence, on first reading, *one may skip the sections on finite state modeling* and study these only after covering the full development of the continuous state models together with the numerical results of the simulations described later in Section 8.4.

The endogenous amplification in Eq. 8.19 induces an expansion of the state space and explains how beliefs increase price volatility above the "fundamental" volatility of dividends.

Under the assumptions of marginal Markov and IID distributions, the empirical distribution of the eight states is characterized by an 8×8 stationary transition matrix M. Hence, this matrix has two important properties: The dividend transition matrix (Eq. 8.10b) must be one marginal probability and the unconditional belief probabilities $Q^i\{g_t{}^i = 1\} = \alpha_i$ must be the second marginal probabilities. Denoting the multiplication $\alpha[M_{v,u}] = [\alpha M_{v,u}]$, it turns out that M must take the form

$$M = \begin{bmatrix} \varphi M(a) & (1-\varphi)M(a) \\ (1-\varphi)M(b) & \varphi M(b) \end{bmatrix} \tag{8.20}$$

where $M(a)$ and $M(b)$ are 4×4 matrices that take the form

$$M(a) = \begin{bmatrix} a_1 & \alpha_1 - a_1 & \alpha_2 - a_1 & 1 + a_1 - \alpha_1 - \alpha_2 \\ a_2 & \alpha_1 - a_2 & \alpha_2 - a_2 & 1 + a_2 - \alpha_1 - \alpha_2 \\ a_3 & \alpha_1 - a_3 & \alpha_2 - a_3 & 1 + a_3 - \alpha_1 - \alpha_2 \\ a_4 & \alpha_1 - a_4 & \alpha_2 - a_4 & 1 + a_4 - \alpha_1 - \alpha_2 \end{bmatrix}$$
$$M(b) = \begin{bmatrix} b_1 & \alpha_1 - b_1 & \alpha_2 - b_1 & 1 + b_1 - \alpha_1 - \alpha_2 \\ b_2 & \alpha_1 - b_2 & \alpha_2 - b_2 & 1 + b_2 - \alpha_1 - \alpha_2 \\ b_3 & \alpha_1 - b_3 & \alpha_2 - b_3 & 1 + b_3 - \alpha_1 - \alpha_2 \\ b_4 & \alpha_1 - b_4 & \alpha_2 - b_4 & 1 + b_4 - \alpha_1 - \alpha_2 \end{bmatrix} \tag{8.21}$$

φ is due to dividends and (α_1, α_2) is due to individual beliefs. This leaves open a and b, which reflect correlation *among* beliefs and *between* dividend growth and beliefs. In an uncorrelated world $a_i = b_i = 0.25$ for all i. If beliefs and dividend growth are uncorrelated $a = b$. Correlation does not arise by individual choices, *hence* (a, b) *reflect the externality of beliefs determined by social interaction and communication.* It is the crucial component of endogenous amplification of beliefs.

Now use Theorem 8.3 and Example 8.2 to define a Markov belief as a conditional probability $Q^i(s|g_t{}^i, H_t), s = 1, 2, \ldots, 8$, on the eight states in Eq. 8.19; implied by a 16×16 matrix F which is a joint probability on the eight economy-wide states and the two individual belief states. By Theorem 8.3 (see also Example 8.2) agent i is using two transition matrices $(F_1{}^i, F_2{}^i)$, with a conditioning as follows:

$$F_v{}^i = Q^i(\bullet | g_t{}^i = v, H_t) \quad \text{if} \quad g_t{}^i = v \quad \text{for} \quad i = 1, 2, \; v = 1, 2$$

and α_i is then the unconditional frequencies at which agent i uses matrix $F_1{}^i$. *Since* M *is deduced from the data, rationality of belief* requires the two pairs of matrices to satisfy the conditions

$$M = \alpha_i F_1{}^i + (1 - \alpha_i) F_2{}^i \quad \text{for } i = 1, 2 \tag{8.22}$$

Keeping in mind that the empirical distribution is given to an agent, rationality implies that relative to M agent i can select only ${F_1}^i$, since Eq. 8.22 implies that ${F_2}^i = \frac{1}{1-\alpha_i}(M - \alpha_i {F_1}^i)$. This imposes additional restrictions on ${F_1}^i$ due to the nonnegativity inequality conditions $M \geq \alpha_i {F_1}^i$. In sum, we have:

Theorem 8.5. *In the case of two types, two states of belief and two state Markov process of the dividend process the set of all rational beliefs relative to M is characterized by the set of* $(0 \leq \alpha_1 \leq 1, 0 \leq \alpha_2 \leq 1)$ *and pairs of nonnegative transition matrices* $({F_1}^i, {F_2}^i)$*, which satisfy Eq. 8.22.*

But what are the matrices of the two agents? Since any two matrices $({F_1}^i, {F_2}^i)$ of agent i are perturbations of M, there is a limited choice of matrices satisfying the inequalities $M \geq \alpha_i {F_1}^i$. In the simulation models of asset volatility cited previously, researchers chose a simple formulation that permits an agent to be either optimistic or pessimistic about the probability of high dividend states tomorrow. To do that, note that the first four rows and columns of M correspond to the high dividend state and the second four rows and columns to the low dividend state. Hence, the 4×4 matrix $\varphi \chi^i M(a)$ would express optimism or pessimism by a factor χ_i, relative to $\varphi M(a)$, in transition probabilities from a high dividend state today to a high dividend state tomorrow. For ${F_1}^i$ to be a transition matrix one must also adjust the matrix $(1 - \varphi)M(a)$.

For a specified parameter $0 \leq \alpha_i \leq 1$ and subject to the nonnegativity restrictions specified earlier, the following matrix is a rational belief that expresses optimism (if $\chi^i > 1$) or pessimism (if $\chi^i < 1$) in transition to high dividend state from all states today:

$$F_1^i = \begin{bmatrix} \varphi \chi^i M(a) & (1 - \varphi \chi^i) M(a) \\ (1 - \varphi)\chi^i M(b) & (1 - (1 - \varphi)\chi^i) M(b) \end{bmatrix} \tag{8.23}$$

In short, for a given $(0 \leq \alpha_1 \leq 1, 0 \leq \alpha_2 \leq 1)$ the set of rational beliefs of this form is summed up by one parameter χ^i, which varies over an interval defined by the inequality $M \geq \alpha_i {F_1}^i$. Now, if $\chi^i > 1$, an agent using ${F_1}^i$ is optimistic about d_{t+1}: his conditional probability of $d_{t+1} = d^H$ is higher than the stationary probability implied by M. When this agent uses ${F_2}^i$, his conditional probabilities of $d_{t+1} = d^H$ are lower than the probability implied by M. The results of these studies are reported later.

8.3.5. Asset Pricing with Heterogeneous Beliefs: An Illustrative Model and Implications

Having outlined the structure of beliefs, we now return to the infinite horizon model of Section 8.2.2 and adapt it to an economy with diverse beliefs but common information. Assume a continuum of agents on $[0, 1]$ and an exogenous risky payoff process $\{d_t, t = 1, 2, \ldots\}$ with an unknown stable and ergodic probability $\prod$ and empirical distribution described by a transition $d_{t+1} = \lambda_d d_t + \rho_{t+1}^d$ and where $\rho_{t+1}^d \sim N(0, \sigma_d^2)$, IID. The asset structure of the economy consists of an aggregate stock index (think of it as the S&P 500) and a risk-free bond. We assume the *riskless rate* r *is constant over time* and positive; hence $R = 1 + r > 1$ and $0 < \frac{1}{R} < 1$. Agent i borrows the amount B_t^i at t and receives with certainty $B_t^i R$ at $t+1$. At date t, agent i buys θ_t^i shares of stock and receives dividend $D_t = d_t + \mu$ for each θ_{t-1}^i held. Consumption is then standard: $c_t^i = \theta_{t-1}^i[p_t + d_t + \mu] + B_{t-1}^i R - \theta_t^i p_t - B_t^i$. Equivalently, we define wealth $W_t^i = c_t^i + \theta_t^i p_t + B_t^i$ and derive the familiar transition for wealth

$$W_{t+1}^i = (W_t^i - c_t^i)R + \theta_t^i \pi_{t+1} \quad \pi_{t+1} = p_{t+1} + (d_{t+1} + \mu) - Rp_t \tag{8.24}$$

π_t is excess returns per share. For initial values (θ_0^i, W_0^i) the agent maximizes the expected utility

$$\underset{(\theta^i, c^i)}{E_t^i} \left[\sum_{s=0}^{\infty} -\beta^{t+s} e^{-(\gamma c_{t+s}^i)} | H_t \right] \tag{8.25}$$

subject to transitions Eqs. 8.17a–8.17c of the state variables $\psi_t^i = (1, d_t, Z_t, g_t^i)$. H_t is date t information.

Our assumptions are restrictive. Constant R is not realistic and exponential utility exhibits no income effects. Nevertheless, these assumptions have the great advantage of leading to closed form solutions that are helpful vehicles to explain the main ideas. Hence the term *illustrative* in this section's title. To seek a closed-form solution we conjecture that prices are linear in the economy's state variables; hence equilibrium price p_t is conditionally normally distributed. In Theorem 8.4 we confirm this conjecture. For an optimum (for details, see the Appendix of Kurz and Motolese, 2007) there exists a constant vector u, so the demand functions for the stock is

$$\theta_t^i(p_t) = \frac{R}{\gamma r \hat{\sigma}_\pi^2}[E_t^i(\pi_{t+1}) + u\psi_t^i], \; u = (u_0, u_1, u_2, u_3), \; \psi_t^i = (1, d_t, Z_t, g_t^i) \tag{8.26}$$

$\hat{\sigma}_\pi^2$ is an *adjusted* conditional variance (the "adjustment" is explained in Section 8.4.3) of excess stock return π_{t+1}, which is assumed to be constant and the same for all agents. The term $u\psi_t^i$ is the intertemporal hedging demand that is linear in agent i's state variables.

For an equilibrium to exist, we impose stability conditions on the dynamics of the economy:

Stability Conditions We require that (i) $0 < \lambda_d < 1$, (ii) $0 < \lambda_Z + \lambda_Z^g < 1$.

(i) requires that $\{d_t, t = 1, 2, \ldots\}$ is dynamically stable, and (ii) requires dynamic stability of *belief*. It requires the market, on average, to believe that (d_t, Z_t) is stable. To see why, look at this definition.

Definition 8.7. The average market belief operator is $\overline{E}_t(\bullet) = \int E_t^i(\bullet) di$.

Now take expectations of Eq. 8.17b, average over the population and recall that Z_t are averages of $g_t{}^i$. This implies that

$$\overline{E}_t[Z_{t+1}] = (\lambda_Z + \lambda_Z^g)Z_t$$

Kurz (2007) and Kurz and Motolese (2007) then demonstrate the following results.

Theorem 8.6. *Consider the model with heterogeneous beliefs under the stability conditions specified with supply of shares equal to 1. Then there is a unique equilibrium price function, which takes the form*

$$p_t = a_d d_t + a_z Z_t + P_0 \tag{8.27a}$$

with coefficients

$$a_d = \frac{\lambda_d + u_1}{R - \lambda_d} \tag{8.27b}$$

$$a_z = \frac{(a_d + 1)\lambda_d^g + (u_2 + u_3)}{R - (\lambda_Z + \lambda_Z^g)} \tag{8.27c}$$

$$P_0 = \frac{(\mu + u_0)}{r} - \frac{\gamma \hat{\sigma}_\pi^2}{R} \tag{8.27d}$$

The linearity of the price thus confirms the earlier conjecture that the price is conditionally normal.

Now, closed-form solutions for the hedging demand parameters $u = (u_0, u_1, u_2, u_3)$ are not available; hence Kurz and Motolese (2007) compute numerical Monte Carlo solutions. For all values of the model parameters they find (i) $a_d > 0$, (ii) $(a_d + 1)\lambda_d^g + (u_2 + u_3) > 0$, and (iii) $a_z > 0$. These conclusions are reasonable: *Today's asset price increases if d_t or Z_t rise.*

Our main objective now is to assess the implications of theory to *the effects of diverse rational beliefs on asset market dynamics*. We do it in two ways. First, we use the closed-form solution of the illustrative model as a simple reference. Second, we use the developments up to now, together with a citation of other papers, to develop results on the questions at hand. We devote the rest of this section to a discussion of such implications of the theory.

Endogenous Uncertainty

The most direct implication of the theory is that asset markets are subject to *endogenous uncertainty*. To explore Definition 8.6, we examine the price map $p_t = a_d d_t + a_z Z_t + P_0$ and find that endogenous uncertainty is expressed in two ways. First the term Z_t says that the risks of asset returns are, in part, due to the risk of future market belief. By Eqs. 8.3.4a and 8.3.4b, in the long run $\sigma_p^2 = a_d^2\sigma_d^2 + a_z^2\sigma_Z^2$; hence price volatility is caused by exogenous as well as endogenous forces, and this has far-reaching implications to market efficiency, risk premia, and public policy. Second, Kurz (2007) shows that the variance in P_0 can be approximated by $\hat{\sigma}_\pi^2 \simeq (a_d+1)^2\hat{\sigma}_d^2 + 2(a_d+1)a_z\hat{\sigma}_{Zd} + a_Z^2\hat{\sigma}_Z^2$. These are terms from the covariance matrix in the agent belief Eqs. 8.17a–8.17c; hence they depend on perception rather than on the actual empirical moments in Eqs. 8.16a–8.16b. But as the perceived volatility of dividend and average market belief increases, the price declines. We show in the next section that this fact implies an increased risk premium.

The presence of endogenous uncertainty in asset markets has far-reaching implications to asset-pricing theory and market dynamics and we note a few of these general conclusions:

- An asset's price is not equal to a unique fundamental value determined by the flow of future payoffs. Moreover, market belief about exogenous states matters since it is often wrong; hence market belief is an independent and dominant component of asset price volatility.
- Moral hazard and the large dimension of market belief make it impossible for markets to trade contracts contingent on market belief; hence markets are fundamentally incomplete.
- In scales of days or weeks, changes in productivity, growth, and profits are slow. Hence, it is absolutely clear without much formal analysis that most volume of trading results from changes in the market distribution of beliefs. Indeed, over the short run the key function of asset markets is to permit agents to *trade their belief differences*.
- Expected individual excess returns and "efficient frontiers" are both subjective concepts. Hence, in markets with diverse beliefs most predictions of CAPM theory do not hold.
- By anonymity of individuals, the market belief is a public externality and hence subject to the effect of coordination and public policy. Stabilization policy can thus have a strong effect on market volatility, and this carries over to monetary economies as well.

The Endogenous Uncertainty Risk Premium

We now turn to an exploration of the risk premium under heterogeneous beliefs in the illustrative model of Section 8.3.5, and we review results in Kurz and Motolese (2007). Recall that the premium on a long position, *as a random variable*, is defined by

$$\frac{\pi_{t+1}}{p_t} = \frac{p_{t+1} + d_{t+1} + \mu - Rp_t}{p_t}. \tag{8.28}$$

We seek a measure of the premium as a *known expected quantity* recognized by market participants, but we have a problem, since with diverse beliefs the premium is subjective. From Eqs. 8.27a–8.27d we compute three equilibrium measures to consider. One is the subjective expected excess returns by i,

$$E_t^i(\pi_{t+1}) = (a_d + 1)(\lambda_d d_t + \lambda_d^g g_t^i) + a_z(\lambda_Z Z_t + \lambda_Z^g g_t^i) + \mu + P_0 - Rp_t \qquad \textbf{(8.28a)}$$

Aggregating over i we define the *market premium* as the average market expected excess returns. It reflects what the market expects, not necessarily what the market gets:

$$\overline{E}_t(\pi_{t+1}) = (a_d + 1)(\lambda_d d_t + \lambda_d^g Z_t) + a_z(\lambda_Z Z_t + \lambda_Z^g Z_t) + \mu + P_0 - Rp_t \qquad \textbf{(8.28b)}$$

Eqs. 8.28a and 8.28b are not necessarily "correct," and we focus on a third, objective, measure, which is common to all. Econometricians who study the long-term time variability of the premium measure it by the empirical distribution of Eq. 8.28, which, by Eqs. 8.27a–8.27d and Eqs. 8.16a and 8.16b, is

$$E_t^m[\pi_{t+1}] == (a_d + 1)(\lambda_d d_t) + a_z \lambda_Z Z_t + \mu + P_0 - Rp_t \qquad \textbf{(8.28c)}$$

Eq. 8.28c is the common way all researchers cited previously have measured the risk premium; therefore we refer to it as *the* risk premium.

We thus arrive at two conclusions. First, the difference between the individual perceived premium and the market perceived premium is

$$E_t^i[\pi_{t+1}] - \overline{E}_t[\pi_{t+1}] = [(a_d + 1)\lambda_d^g + a_z \lambda_Z^g](g_t^i - Z_t) \qquad \textbf{(8.29a)}$$

From the perspective of trading, all that matters is the difference $g_t^i - Z_t$ between individual and market belief. In addition, the following difference is important:

$$E_t^m[\pi_{t+1}] - \overline{E}_t[\pi_{t+1}] = -[(a_d + 1)\lambda_d^g + a_z \lambda_Z^g]Z_t \qquad \textbf{(8.29b)}$$

The risk premium is different from the market perceived premium when $Z_t \neq 0$. But the important conclusion is the analytical expression of the risk premium:

$$E_t^m[\pi_{t+1}] = \left(\frac{\gamma r \hat{\sigma}_\pi^2}{R} - u_0 - u_1 d_t\right) - a_z(R - \lambda_Z)Z_t \qquad \textbf{(8.30)}$$

Since $a_z > 0$, $R > 1$, and $\lambda_Z < 1$, it follows that the premium per share *declines* with Z_t. We then have Theorem 8.7.

Theorem 8.7. *The risk premium $E_t^m[\pi_{t+1}]$ is increasing in the variance $\hat{\sigma}_\pi^2$ and decreasing in the mean market belief Z_t.*

This theorem exhibits what Kurz and Motolese (2007) call the "Market Belief Risk Premium." It shows that the risk premium depends on market belief in two ways:

1. A direct effect on the permanent mean premium $\frac{\gamma r \hat{\sigma}_\pi^2}{R}$. We have seen that the variance is approximately $\hat{\sigma}_\pi^2 \simeq (a_d + 1)^2 \hat{\sigma}_d^2 + 2(a_d + 1)a_z \hat{\sigma}_{Zd} + a_Z^2 \hat{\sigma}_Z^2$; hence it increases with the perceived volatility of dividend and the volatility of average market belief.

2. An effect on the time variability of the risk premium, expressed by $-a_z(R - \lambda_Z)Z_t$ with a negative sign when $Z_t > 0$.

To understand the second result, note that it says if we run a regression of excess returns on the observable variables, the effect of the market belief on excess return *is negative*. From an REE perspective this sign is somewhat surprising, since when $Z_t > 0$ the market expects *above-normal* future dividends but instead, the risk premium on the stock *declines*. When the market holds bearish belief about dividends ($Z_t < 0$), the risk premium *rises*. This requires some further explanation.

Why is the effect of Z_t on the risk premium *negative*? The result shows that when the market holds abnormally favorable belief about future payoffs of an asset, the market views the long position as less risky and the risk premium on the long position of that asset falls. Fluctuating market belief implies time variability of risk premia, but fluctuations in risk premia are inversely related to the degree of market optimism about future prospects of asset payoffs.

To further explore the result, it is important to explain *what it does not say*. One might interpret it as confirming a common claim that to maximize excess returns it is an optimal strategy to be a "contrarian" to the market consensus by betting against it. To understand why this is a false interpretation, note that when an agent holds a belief about future payments, the market belief does not offer any new information to alter the individual's belief about the exogenous variable. If the agent believes that future dividends will be abnormally high but $Z_t < 0$, the agent does not change her forecast of future dividends. She uses the market belief information only to forecast future *prices* of an asset. Thus, Z_t is a useful input to forecasting returns without changing the forecast of d_{t+1}. Given the available information, an optimizing agent is already placed on her demand function defined relative to her own belief; hence it is not optimal for her to just abandon her demand and adopt a contrarian strategy.

This argument is the same as the one showing why it is not optimal to adopt the log utility as your own utility, even though it maximizes the growth rate of your wealth. Yes, it does, but you dislike the sharp expected declines in the values of your assets. By analogy, following a "contrarian" policy may imply a high long-run average return in accord with the empirical probability m. However, if you disagree with this probability, you will dislike being short when your true optimal position is to be long. Indeed, this argument explains why *most people hold positions that are in agreement with the market belief most of the time* instead of betting against it. The crucial observation to make is that a maximizing agent has his own belief about future events, and he does not select a new belief when he learns the market belief. From his point of view, the market belief is an important state variable used to forecast future prices. When it is wrong, the market may forecast a recession that never arrives.

Theorem 8.7 was derived for an exponential utility function. Kurz and Motolese (2007) show that this result is more general and depends only on the positive coefficient a_z of Z_t in the price map. For more general utility functions, they use a linear approximation to show that the result depends only on the condition that the slope of the stock price is positive with respect to Z_t. This condition requires the current stock price to increase if the market is more optimistic about the asset's future payoffs.

Finally, Kurz and Motolese (2007) use data compiled by the Blue Chip survey of forecasts to test the theory proposed in Eq. 8.35b. They report that the data support the theoretical results.

Rational Overconfidence

Evidence from the psychological and behavioral literature (e.g., Svenson, 1981; Camerer and Lovallo, 1999; and Russo and Schoemaker, 1992) shows a majority of individuals assess their own probability of success in performing a task (investment, economic decisions, driving, etc.) above the empirical frequency of success in a population. Hence a majority of people often expect to outperform the empirical frequency measured by the median or mean. In a *rational expectations* paradigm, individuals know the true probability of success; hence the observed inconsistency is taken to be a demonstration of irrational behavior. Indeed, inconsistency between individual assessments and empirical frequencies has been cited extensively as a "proof" of irrational behavior and in support of behavioral/psychological impulses for belief and forecasting. This phenomenon has thus been called *overconfidence*. We reject this conclusion and show that it reveals a fundamental flaw.

The work cited previously (and other empirical and experimental work) provides evidence against *rational expectations*. But rational expectations is an extreme theory in demanding agents to know the full structure of the economy and make exact probability assessments. Behavioral economics takes the other extreme view and assumes that people are irrational and motivated by psychological impulses. Hence, a rejection of rational expectations does not imply acceptance of irrationality of agents. Indeed, we may reject these two extreme perspectives by observing the fact that most people do the best they can, given the limited knowledge they have. Rational people do not know everything and make "mistakes" relative to a true model they do not know. The theory of *rational beliefs* rejects both extremes in favor of an intermediate concept of rationality. We then show that *overconfidence* is compatible with *rational beliefs* and, indeed, *agents who hold rational belief will universally exhibit "rational overconfidence."* Hence, the cited empirical evidence is no proof that people are irrational and motivated by pure psychological factors.

We explain the preceding by using Example 8.1 (also, see Nielsen, 2006). A group of gamblers look at the black box in Example 8.1 and form beliefs using different sequences $g = (g_0, g_1, \ldots)$ as in that example. Each belief is then defined by a sequence of *independent* random variables satisfying Eq. 8.9a. Gamblers vary with the sequences g they use. A survey is taken and the distribution of beliefs is publically announced. Hence, belief distributions of past gamblers are known but not their individual beliefs. Since $(1/3)(0.60) + (2/3)(0.45) = 0.50$, all are rational beliefs for almost all g. In the *rational beliefs* literature the ratio 1/3 is referred to as the "frequency" of optimism. When the frequency of bull and bear states is not the same, we have a *market asymmetry* between them. The probability 0.60 is the "intensity" of optimism when optimistic. In defining a *rational belief* these characteristics are selected separately: For each frequency there is a range of feasible intensities that are rational.

The gamblers decide at date $t-1$ how they want to bet. They can gamble \$1 on $v_t = 1$ or on $v_t = 0$: They win \$1 if they are right and they lose \$1 if they are wrong. Since it's a small bet, they will all bet. Those who put money on 1 expect to win with a probability 0.60; those who put their money on 0 expect to win with probability 0.55. *They are all overconfident and rational!* The constancy of the high (0.60) and low (0.45) probabilities is not essential since we can, instead, put in any sequences of parameters that converge to 0.50 from above and from below and the result will be the same.

Observe that in this example all deviations from empirical frequencies lead to optimal behavior that exhibits *universal* overconfidence. When a subjective probability is above the empirical frequency, a long position is optimally taken with overconfidence. When a probability is below the empirical frequency, a short position is optimally held with overconfidence. *Hence, all agents are then optimally overconfident at all times.*

Generalizing the example is natural. Beliefs are all about deviations from empirical frequencies on the basis of which economic decisions are made. Optimistic agents engage in taking the risk of success in an activity, and pessimistic agents engage in gambles against success. If they cannot gamble against it (e.g., short positions are not allowed), they refrain from participation. This type of behavior is then natural to the *rational belief* paradigm. Moreover, this behavior is natural to any complex environment in which aggregation of subjective probability beliefs of agents may not be equal to the empirical frequencies. But then all creative work and all innovative decisions are based on beliefs that exhibit "overconfidence." Indeed, one can hardly think of entrepreneurship, inventive activity, and any speculative behavior without beliefs that exhibit rational overconfidence.

Properties of Average Market Belief and Higher-Order Beliefs

By Eqs. 8.17a and 8.17b and Definition 8.7 it follows that:

$$\overline{E}_t \begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} - E_t^m \begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_d^g Z_t \\ \lambda_Z^g Z_t \end{pmatrix} \tag{8.31}$$

and Eq. 8.31 exhibits the dynamics of the average market belief operator. However, Eqs. 8.17a–8.17c also show that properties of conditional probabilities do not apply to the market belief operator $\overline{E}_t(\bullet)$ since it is *not a proper conditional expectation.* To see why, let $X = D \times Z$ be a space where (d_t, Z_t) take values, and let G^i be the space of g_t^i. Since i conditions on g_t^i, his unconditional probability is a measure on the space $((D \times Z \times G^i)^\infty, \mathcal{F}^i)$, where $\mathcal{F}^i$ is a sigma field. The market conditional belief operator is just an average over conditional probabilities, each conditioned on a *different* state variable. Hence, this averaging does not permit one to write a probability space for the market belief. *The market belief is neither a probability nor rational!* This is then formulated (see Kurz, 2007) as Theorem 8.8.

Theorem 8.8. *The market belief operator violates iterated expectations:* $\overline{E}_t(d_{t+2}) \neq \overline{E}_t\overline{E}_{t+1}(d_{t+2})$.

Our earlier comment about the importance of the treatment of market belief as a state variable with independent dynamics is now complemented by Theorem 8.8. Market belief is an externality that does not arise from rational social choice of a collective agent. It cannot arise in a model of intertemporal choice of a single representative agent.

Turning to higher-order beliefs, we must distinguish between higher-order beliefs that are *temporal* and those that are *contemporaneous.* Eqs. 8.17a–8.17c define agent i's belief over future sequences of (d_t, Z_t, g_t^i) and as is the case for any probability, it implies i's temporal higher-order belief with regard to future events. For example, we deduce from Eqs. 8.17a–8.17c statements such as:

$$E_t^i(d_{t+N}) = E_t^i E_{t+1}^i \dots E_{t+N-1}^i(d_{t+N}), \quad E_t^i(Z_{t+N}^i) = E_t^i E_{t+1}^i \dots E_{t+N-1}^i(Z_{t+N}^i) \quad \textbf{(8.32)}$$

Properties of temporal higher-order beliefs are thus familiar properties of conditional expectations.

As to market belief, since Eq. 8.15 is implied by Eq. 8.17c, the average market belief operator satisfies $\overline{E}_t(d_{t+N+1}) = \lambda_d \overline{E}_t(d_{t+N}) + \lambda_d^g \overline{E}_t(Z_{t+N})$. We deduce perceived higher-order temporal market beliefs by averaging over i. For example,

$$\lambda_d^g \lambda_Z \overline{E}_t(Z_{t+N}) = \overline{E}_t\overline{E}_{t+N+1}(d_{t+N+2}) - \overline{E}_t E_{t+N+1}^m(d_{t+N+2}) \qquad \textbf{(8.33)}$$

Contemporaneous higher-order beliefs have attracted attention (e.g., Allen, Morris, and Shin, 2006; Bacchetta and van Wincoop, 2005; and Woodford, 2003) despite being unobservable. They occur naturally in strategic situations. In a market context they can *formally* arise in Eqs. 8.17a–8.17c as follows: Let Z_t in Eq. 8.15 be defined as Z_t^1. We can argue that agents may form beliefs about the future of this variable by using a *second* belief index g_t^{i2} about Z_{t+1}^1 whose transition would be deduced from the transition of g_t^{i2}. Now $Z_t^2 = \int g_t^{i2} di$ would be a second-order aggregate belief for which a *third* belief index g_t^{i3} could be introduced whose average would be Z_t^3, and so on. Such infinite regress is problematic and leads us to reject contemporaneous higher-order beliefs in markets for two reasons. First, higher-order beliefs are degenerate in Eqs. 8.17a–8.17c *because* the single-belief index g_t^i fully pins down agent i's belief. Moreover, since agents know the beliefs of others and all variables in the price map (which embody the beliefs of others), there is nothing else about which to form beliefs. There is a second and more general reason why, in markets, *all higher-order beliefs* Z_t^j, *for* $j > 1$ *are degenerate.* This is so since they are averages of g_t^{ij} and since for $j > 1$ the Z_t^j are not observable; they can only exist in the minds of the agents, and hence there is no possible mechanism for individual g_t^{ij} to be correlated as in Eq. 8.15. Hence, higher-order beliefs cannot have an aggregate effect, since with independent g_t^{ij} the averages Z_t^j for $j > 1$ are zero at all t.

On Beauty Contests

The Keynes Beauty Contest metaphor has been extensively discussed. Some have associated it with asset-pricing equilibrium, where the price is expressed as iterated expectations of average market belief of the future *fundamental value* of the asset. In Eq. 8.8 we presented the Allen Morris and Shin (2006) example of such pricing with private information. But this interpretation should be questioned. An examination of Keynes' view (see Keynes, 1936, page 156) shows that the crux of Keynes's conception is that there is little merit in using fundamental values as a yardstick for market valuation. Hence what matters for the asset demand of an agent is the perception of what the market believes the future price of that asset will be rather than what the intrinsic value is. Keynes insists that future price depends on future market belief and that *may be right or wrong* without a necessary relation to an intrinsic fundamental value. The Beauty Contest parable is thus simple: A price does not depend on an intrinsic value but on what the market believes future payoffs and valuations will be. Keynes's Beauty Contest is thus a statement that to forecast the price in the future, an individual must forecast the future market state of belief, *when such forecasts may be "right" or "wrong."* We now observe that a *rational belief equilibrium* captures the essence of the Keynes Beauty Contest.

To explain, we make two observations. From Eq. 8.27a, equilibrium price is $p_t = a_d d_t + a_z Z_t + P_0$, and this is clearly in accord with the preceding: *In any model of the Beauty Contest, equilibrium price should not depend on a true intrinsic value; rather, it should depend on market belief.* It follows from the rationality conditions that price/earning ratios exhibit fluctuations with reversion to the long-run stationary mean, but such long-term value is not an intrinsic fundamental value. Indeed, in a model with diverse belief there is no such thing as fundamental intrinsic value, since all prices depend on market belief. Second, to forecast future prices, an agent forecasts Z_{t+1}, which is the market belief tomorrow. From Eq. 8.17b, we have $Z^i_{t+1} = \lambda_Z Z_t + \lambda^g_Z g^i_t + \rho^{iZ}_{t+1}$, which means that an agent forecasts the future market belief with his own subjective model. In sum, this equilibrium concept reflects the Beauty Contest parable because the price map depends on market beliefs, not on some agreed-on intrinsic value, and to forecast future prices agents must forecast the belief of others.

Speculation

Although market practitioners have an intuitive idea of what "speculation" is, there is no scientific consensus on how to define this concept. Keynes (1936) viewed asset markets as a "beauty contest," and many writers have interpreted this to be a form of speculation. A different perspective was proposed by Kaldor (1939), who defined speculation as "the purchase (or sale) of goods with a view to resale (repurchase) at a later date." It is clear that for such asset trades to make sense, prices of assets must regularly deviate from their fundamental values, and agents must believe that prices, are or will not be equal to their fundamental values. It is also clear that in a perfect REE world with homogenous beliefs and complete information, a Kaldor speculation is not possible (e.g., see Tirole, 1982; Milgrom and Stokey, 1982). Here we explore the perspective of a diverse belief equilibrium with respect to Kaldor (1939) speculation.

Following the definition of Kaldor (1939), Harrison and Kreps (1978) study the consequences of risk-neutral investors having different beliefs about the dividend process of a risky asset. At date t, investor i can expect a payment $E_t^i(\beta^k p_{t+k} + \sum_{s=0}^{k} \beta^s (d_{t+s} + \mu))$ if he chooses to resell k periods later, where $\{p_t\}$ and β denote the stock price process and the discount rate. The equilibrium market price, called a *consistent price scheme*, is the supremum over all stopping times k and across all investors. That is, this price is

$$p_t = \max_i \sup_k E_t^i \left(\beta^{t+k} p_{t+k} + \sum_{s=0}^{k-1} \beta^{t+s} (d_{t+s} + \mu) \right)$$

Agents hold diverse beliefs and are assumed to have infinite wealth for each class of investor type. A speculative premium is then defined to be the difference between the consistent price scheme and the value, $\max_i E_t^i(\sum_{s=0}^{\infty} \beta^{t+s}(d_{t+s} + \mu))$, expected when all investors are obliged to hold the asset forever. Harrison and Kreps (1978) show that under the assumptions made, there exists a positive speculative premium, or a price bubble, when short sales are not allowed.

Morris (1996) further examines asset pricing during initial public offerings when investors have different prior distributions, but the difference of belief disappears as investors learn from observations. A major weakness of both the works of Harrison and Kreps (1978) and Morris (1996) is their assumption of the unrestricted heterogeneity of beliefs.

Wu and Guo (2003) use the theory of rational belief to explain the persistence of diverse beliefs in the Harrison and Kreps (1978) model and to narrow the equilibrium results. They adopt the finite state Markov assumption for dividend and the two state of belief model. Wu and Guo (2003) then show that in contrast with the complex solution of Harrison and Kreps (1978) a rational belief equilibrium price vector (over states) is a simple expression that is computed via a finite algorithm. As to dynamics, they show that speculative bubbles and endogenous uncertainty emerge. They further characterize how the speculative premium increases with the degree of heterogeneity.

To explore the phenomenon of *simultaneous* increase in asset prices and trading volume, Wu and Guo (2004) study a model of heterogeneous rational beliefs held by a continuum of agents on the unit interval, as in Miller (1977). In contrast with Harrison and Kreps (1978) and Morris (1996), Wu and Guo (2004) permit limited short sales and impose a wealth constraint of a finite investment fund. They assume an IID dividends process over two states and arrange investors in the order of their optimism along the unit interval. They then derive a steady-state rational belief equilibrium price and show that in equilibrium optimistic investors hold the entire supply. In this framework, Wu and Guo (2004) demonstrate the emergence of endogenous uncertainty with a positive speculative premium that increases with the size of the investment fund and degree of optimism and decreases with the size of short-sale constraint. Furthermore, the model generates a positive relationship between trading volume and the directions of price changes: Volume is high when prices rise and it is low when prices decline. There is also a positive relationship between trading volume and price level. These results are consistent with the empirical evidence (e.g., Karpoff, 1987, and Basci et al., 1996).

8.4. EXPLAINING MARKET DYNAMICS WITH SIMULATION MODELS OF DIVERSE BELIEFS

Although much of our discussion is analytical, significant results about excess volatility are deduced from simulation models. Simulations require specification of functions, parameters, and beliefs and aim to show that a model replicates the statistics of the economy.

8.4.1. Introduction: On Simulation Methods and the Main Results

Since most models reviewed have only two agent types, the beliefs selected are representative of only two classes of agent. Since we might question the validity of such an approach as too narrow, it is useful to explain the *common features* of all simulation models reviewed that replicate the dynamics of real markets we observe. Our view on this issue is simple: a simulation model is a very good tool to explore the impact of the *qualitative features* of feasible belief structure on market volatility. The specific parameter configurations used to attain these qualitative features are less important.

The best way to explain this view is to highlight the central conclusions of the work we review in this section, and the summation of Kurz et al. (2005a) is useful. This paper starts from the view that in any non-REE-based asset market theory there are basically two *natural individual states*: optimistic (i.e., bull states) and pessimistic (i.e., bear states). The authors then explain that given these two basic states, there are three central characteristics of individual beliefs that fully account for all characteristics of market volatility and risk premia observed in real markets. These are:

1. Large (i.e., high-intensity) fat tails in the belief densities of agents
2. Asymmetry in the proportion of bull and bear states in the market over time
3. Belief states are correlated, resulting in regular joint dynamics of belief distributions

Large fat tails means that the densities of the agents' beliefs have very fat tails. "Intensity" measures size of deviations from stationary probabilities, as in the review of rational overconfidence. The *asymmetry in the time frequency of belief states* needed to reproduce the results is a subtle feature that says that on average, agents are in bear states at more than 50% of the dates. Equivalently, on average, at more than half the time, agents do not expect long positions to make *above-normal* returns on their investments. Therefore, it follows from the rational belief principle that when agents are in bull states and expect *above-normal* returns, their expected excess returns must be very high. We shall see later that this asymmetry is empirically supported by the fact that major abnormal rises in stock prices occur over a relatively small fraction of time. Hence, when agents believe a bull market is ahead, they expect to make excess return in relatively short periods.

Correlation of beliefs is a market externality, not determined by individual choices, which regulates the probability of agreement or disagreement of beliefs in the market and the transitions among such states. This is central because the distribution of beliefs determines prices and returns, and the dynamic of the belief distribution is crucially affected by the correlation. It is then natural that asymmetry in the transitions is important, since it regulates the dynamics of bear vs. bull markets.

Kurz et al. (2005a) then make two observations. First, exactly the same simulation model used to study market volatility is also used to study all other aspects of market dynamics. In that model, stock prices and returns exhibit a structure of forecastability observed in the real data. Also, the same model implies that market returns exhibit stochastic volatility generated by the dynamics of the market beliefs. Second, examination of alternative configurations of belief shows that no other configuration of qualitative features than the three previously specified yields predictions that *simultaneously replicate the empirical record.* Many feasible model parameters generate volatility of prices and returns, but as we move away from the three features, the model fails to generate some essential components of the empirical record, frequently the riskless rate and the risk premium. Thus, the main reason the models are able to explain the empirical record is that they have the needed configuration of qualitative factors. In each case they imply a unique parameter structure of the computational model needed to explain the empirical record, but the specific implied belief is not of central significance.

8.4.2. Anatomy of Market Volatility

Papers on excess volatility simulate computed equilibria with finite or infinite belief states. Those with finite belief states are OLG models, whereas those with infinite belief states are infinite horizon models. This division guides our review.

Understanding the Parametrized Structure of Beliefs

The papers that fall into the first category (that is, OLG) include Nielsen (1996, 2003, 2005, 2006), Kurz (1997b), Black (1997, 2005), Kurz and Beltratti (1997), Kurz and Motolese (2001), Kurz and Schneider (1996), Motolese (2003), Nakata (2007), and Wu and Guo (2003). Papers using infinite horizon models with infinite belief states include Kurz et al. (2005a, 2005b), Kurz (2007), Kurz and Motolese (2007), and Guo and Wu (2007).

We start by discussing OLG models *with finite belief states* and use the parametrization of Kurz and Motolese (2001) as a prototype.[12] All models have two assets: a stock and a riskless bond.

[12]Here again we present the parametrization of finite state and continuous state models. The empirical results reported later are all deduced from the continuous state model; hence it may be useful for the reader to skip, on first reading, material that pertains to finite state models. This material would be especially relevant if the reader wants to replicate any of these results by visiting the Web pages provided in footnotes 13 and 15 to download the programs with which to compute the solutions. It will be found that the finite state models are much easier to handle.

The stock pays dividends with a two-state growth rate. There are two types of agent, each living two periods with a power utility function of agent i over consumption:

$$u(c_t^1, c_{t+1}^2) = [1/(1-\gamma)](c_t^1)^{1-\gamma} + [\beta/(1-\gamma)](c_{t+1}^2)^{1-\gamma}, \gamma > 0, 0 < \beta < 1$$

The belief structure is as in Eqs. 8.20–8.23 with two states; thus there are eight economy-wide states.

Rational beliefs are represented by the two pairs of matrices (F_1^i, F_2^i) with frequencies (α_1, α_2). Since deviations from the long-term mean growth rate of dividends could be either *above* it or *below* it, at each date an agent must be either a bull or a bear about future growth of dividends. This is expressed by a pair of parameters (χ^1, χ^2) that measure *the intensity of optimism when in an optimistic state*, while (α_1, α_2) measure *the frequency of each of the two agents being optimistic.*

To see the implications, note that χ^i are revisions of the probabilities of states (1, 2, 3, 4) and (5, 6, 7, 8) relative to M. $\chi^i > 1$ imply increased probabilities of (1, 2, 3, 4) in matrices F_1^i when the first four prices occur at $d_t = d^H$ states. But these are actually states of high prices as well; hence $\chi^i > 1$ implies that agent i is optimistic about high prices at $t+1$. In all simulations, $\chi^i \geq 1$; hence one interprets g_t^i so that $g_t^i > 0$ means agent i is optimistic (*relative to M*) at t about *high prices* at $t+1$.

In the transition M, the matrices $M(a)$ and $M(b)$ regulate the correlation across beliefs and the effect of dividends on that correlation. This is a *correlation externality* given to agents, which is the same as the correlation among the ρ_t^{ig} across i in Eq. 8.11, a correlation that gives rise to the dynamics of the aggregate Z_t in Eq. 8.15. The correlation is crucial, but it turns out that it does not need to be complex. The case $\chi^1 = \chi^2 = 1$, $\alpha_1 = 0.50$, $\alpha_2 = 0.50$, and $a_i = b_i = 0.25$ is the case of REE. Kurz and Motolese (2001) postulate a simple model with $M(a) = M(b)$; hence beliefs are not correlated with dividends. However, beliefs are correlated with a simple description of $a = b = (.50, .14, .14, .14)$. This simple parametrization implies that the dynamics of prices have the feature that bull and bear markets are *asymmetric*. For the market to transit from the "crash" state of the lowest price to the states of the highest prices, it must take several steps: It cannot go *directly* from the low to the high prices. The opposite is possible, since at the bull market states there is a positive probability of reaching the crash states in one step. *This implies that a bull market that reaches the high prices must evolve in several steps, but a crash can occur in one step.*

To sum up, there are three classes of parameters, and simulation work explores *only regions of the parameter space that are compatible with rationality.* Kurz and Motolese (2001) report a set of parameters under which the model replicates the empirical record with great accuracy. These are: utility function parameters: $2.00 \leq \gamma \leq 3.00$, the common risk aversion coefficient; $0,90 \leq \beta \leq 0.95$, the common discount rate; correlation parameters: $a_1 = 0.50$, $a_2 = a_3 = a_4 = 0.14$; belief parameters: $\chi^1 = \chi^2 = 1.7542$, the maximal intensity permitted by rationality; and $\alpha_1 = \alpha_2 = 0.57$, the frequency with which an agent is in an optimistic state. The model replicates well the empirical record, and the results are similar to those of infinite horizon models. Hence there is no point repeating the same results twice, and we report the precise numerical results only for the infinite horizon models[13]. We thus turn to models with infinite belief states.

Models with infinite horizon and infinite belief states typically have two assets: A stock paying dividends and a zero net supply bond. The model has a large number of identical agents of two types with the same utility and endowment. Across types they differ in their beliefs. For consistency we use a model developed by Kurz et al. (2005a) to illustrate the belief structure and report results for this simulation model. This has the merit that all simulation results reported *are derived from a single model.* These authors assume $D_{t+1} = D_t e^{v_{t+1}}$ with an empirical distribution of the growth rate

$$v_{t+1} = (1 - \lambda_v)v^* + \lambda_v v_t + \rho^v_{t+1} \quad \rho^d_{t+1} \sim N(0, \sigma^2_d)\text{IID}$$

where v^* is the unconditional mean. Given his belief, agent i maximizes an infinite horizon expected utility with date t utility of $\beta^t[1/1-\gamma](c^i_t)^{1-\gamma}$.

To explain the perception models of the agents, we could have postulated $v^i_{t+1} = (1-\lambda_v)v^* + \lambda_v v_t + \lambda^g_v g^i_t + \tilde{\rho}^{v^i}_{t+1}$ as in Eq. 8.17a. Such a model is sufficient for the conceptual needs of the illustrative model, but it would contradict the finite state model reviewed earlier. The reason is that the belief state g^i_t is a *symmetric variable* that does not meet two of the three principles advocated earlier, namely the condition of *asymmetry* and *fat tails* in the belief densities. Limited space permits us to present only a sketch of the complex structure in Kurz et al. (2005a)[14]. To introduce asymmetry and fat tails, the procedure we follow is to transform the g^i_t into a new random variable $\eta^i(g^i_t)$ and define the perception model for the growth rate of dividend to be (8.30)

$$v^i_{t+1} = (1-\lambda_v)v^* + \lambda_v v_t + \lambda^g_v \eta^i_{t+1}(g^i_t) + \tilde{\rho}^{v^i}_{t+1} \quad \tilde{\rho}^{v^i}_{t+1} \sim N(0, \tilde{\sigma}^2_v)\text{IID}$$

We then look for asymmetry and fat tails in the density of $\Delta^i_{t+1}(g^i_t) = \lambda^g_v \eta^i_{t+1}(g^i_t) + \tilde{\rho}^{v^i}_{t+1}$, *conditioned* on g^i_t. Keep in mind that for a computational model, we must choose a specific functional form, and this may appear to be too strong a set of assumptions about beliefs.

[13]For details of the finite belief state results, see Kurz and Motolese (2001, pp. 530–533). For computational procedures to reproduce these results, go to www.stanford.edu/~mordecai/ and click "computable models with heterogeneous beliefs." Keep in mind that an OLG model has a unique market-oriented feature not shared by an infinite horizon model that requires an agent to sell his position when old, regardless of his beliefs. This feature is important for a model of market volatility, since agents who aim to preserve capital by holding a portfolio of a riskless asset must sell the asset into the market, regardless of their beliefs. This fact tends to generate additional volatility that would not be present in an infinite horizon model. This feature has two results that are not shared by the infinite horizon model. First, the riskless rate has a much larger standard deviation in simulated OLG equilibria than in the infinite horizon models. Second, to generate a low average riskless rate, a result needed to replicating the 6–7% equity premium, it is necessary to assume an asymmetry where the majority of agents are optimists about earning abnormal excess returns and the frequency of optimism is greater than 50%. In the infinite horizon model, it is necessary to have the pessimists in the majority, with a frequency of pessimism being more than 50%. We shall comment on this issue again later, when we discuss the Equity Premium Puzzle.

[14]Indeed, the main deficiency of the Kurz et al. (2005a) model is its complexity, which, in our view today, could have been avoided. Both the model itself as well as the computational procedures could have been drastically simplified, since the basic ideas are rather simple, as explained in Section 8.4.1.

How do we know that belief densities take these specific forms? We do not. But here we return to the three *qualitative properties* discussed in Section 8.4.1. What drives the results are not the functional forms selected for a computational model but their qualitative properties. Any other functional form with the same qualitative properties would generate the same results but, naturally, with different parametrization. Since $\eta^i(g_t^i)$ cannot be a simple symmetric variable, Kurz et al. (2005a) specify the conditional distribution of $\eta_{t+1}^i(g_t^i)$. They define it as a step function:

$$P(\eta_{t+1}^i | g_t^i) = \begin{cases} \varphi_1(g_t^i)[1/\sqrt{2\pi}]e^{-\frac{\eta_{t+1}^{i\,2}}{2}} & \text{if } \eta_{t+1}^i \geq 0 \\ \varphi_2(g_t^i)[1/\sqrt{2\pi}]e^{-\frac{\eta_{t+1}^{i\,2}}{2}} & \text{if } \eta_{t+1}^i < 0 \end{cases}$$

where η_{t+1}^i and $\tilde{\rho}_{t+1}^{v^i}$ (in Eq. 8.30) are independent. The functions $(\varphi_1(g), \varphi_2(g))$ are defined by a logistic function with two parameters κ and Λ:

$$\varphi(g^i) = \frac{1}{1 + e^{\Lambda(g^i - \kappa)}}, \kappa < 0, \Lambda < 0 \quad \text{and} \quad \varphi_1(g^i) = \frac{\varphi(g^i)}{E_g\varphi(g^i)}, \varphi_2(g^i) = 2 - \varphi_1(g^i)$$

The parameter κ measures *asymmetry*, and the parameter Λ measures *intensity of fat tails* in beliefs. When $g_t^i > \kappa$, then $E_i[\eta_{t+1}^i | g_t^i] > 0$. Choosing $\lambda_v^g > 0$ implies that when $g_t^i > \kappa$ agent i is in a *bull state* and is optimistic about $t + 1$ dividend growth being above-normal. Since $\kappa < 0$, it also implies that bull states occur with frequency higher than 50%. "Normal" is defined relative to the empirical forecast. In sum, for the basic case $\lambda_v^g > 0$:

- $g_t^i > \kappa$ means that agent j is optimistic about profit growth and excess stock returns at $t + 1$.
- $g_t^i < \kappa$ means that agent j is pessimistic about profit growth and excess stock returns at $t + 1$.

The parameter κ measures asymmetry and determines the frequency at which agents are bears, and when $\kappa < 0$ the probability of $g^i > \kappa$ is more than 50%. The density of η_{t+1}^i is exhibited in Figure 8.4 and shows that asymmetry arises from a redistribution of the probability mass. However, the empirical distribution of $\eta_{t+1}^j(g_t^j)$ averaged over time and over the g_t^i is Normal.

Each component of $\Delta_{t+1}(g_t^i)$ is a sum of two random variables: one as in Figure 8.4 and the second is normal. In Figure 8.5 we draw two densities of $\Delta_{t+1}(g_t^j)$, each being a convolution of the two constituent distributions, one density for $g^i > \kappa$ and a second for $g^i < \kappa$, showing both have "fat tails." Since Λ measures intensity by which the positive portion of the distribution in Figure 8.4 is shifted, it measures the degree of fat tails in the distributions of $\Delta_{t+1}(g_t^j)$.

The assumption of a power utility $\beta^t[1/1 - \gamma](c_t^i)^{1-\gamma}$ implies that income effects matter and beliefs do not aggregate. Hence, the state variable in the simulation model is the actual distribution of beliefs. Since there is a large number of identical agents of *two types*, this distribution is a vector (z_t^1, z_t^2). The fact that we denote it by (z_t^1, z_t^2) and

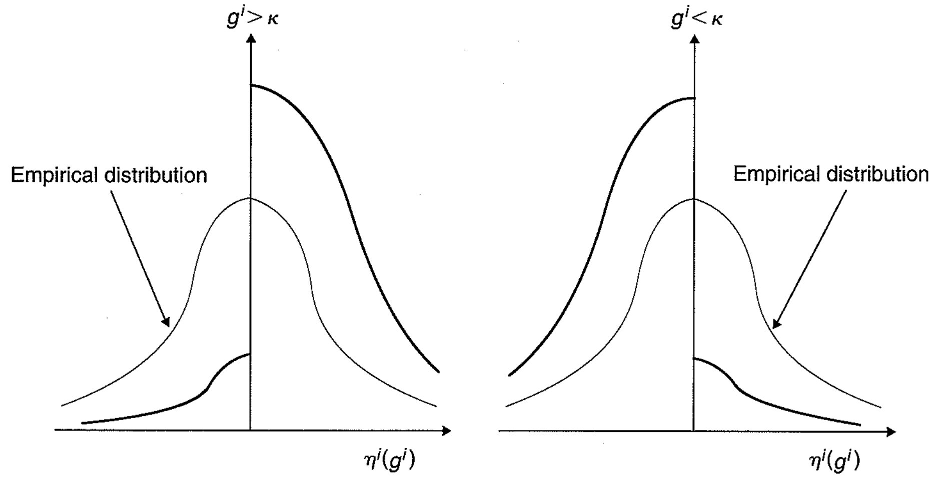

FIGURE 8.4 Nonnormal belief densities.

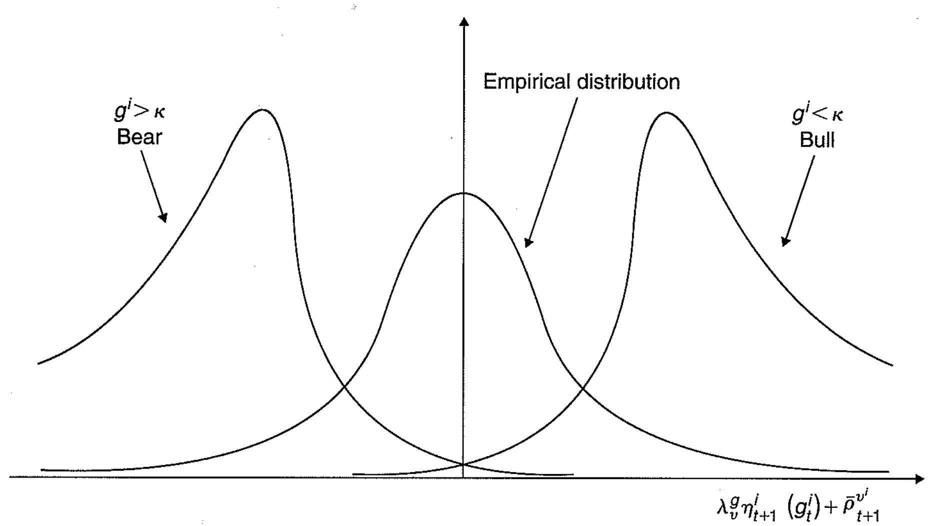

FIGURE 8.5 Density $\Delta_{t+1}^i(g_t^i)$ with fat tails.

not (g_t^1, g_t^2) is an important technical issue arising due to the assumption of anonymity. Agent i knows his own belief as g_t^i, which he uses to forecast all state variables in the economy, whereas (z_t^1, z_t^2) are observed state variables and the agent uses g_t^i to forecast v_{t+1} as in Eq. 8.30 and (z_{t+1}^1, z_{t+1}^2) with a fully developed perception model that is analogous to Eqs. 8.17a–8.17c in the illustrative model we have developed here. For technical details of the perception model and the implied rational belief restrictions, see

Kurz et al. (2005a, pp. 12–19). We now turn to a detailed examination of the simulation results of the finite and infinite belief state model.

Explaining the Volatility Moments

We report simulation results of models with infinite belief states. *All results reported in this chapter are derived from a single model by Kurz et al.* (2005a). In these simulations Kurz et al. compute various measures of volatility using 20,000 observations. Raw moment calculations were carried out by Kurz et al. (2005a) for the following list of long-term volatility measures, which are moments in accord with the stationary measure:

- q: Average price/dividend ratio
- σ_q: Standard deviation of the price/dividend ratio q_t
- R^q: Average risky return on equity
- σ_R: Standard deviation of R^q
- r: Average riskless rate of interest
- σ_r: The standard deviation of r
- e_p: Average the equity premium
- $\rho(d, R^q)$: Correlation between the risky rate and dividends' growth; it is also the correlation between consumption growth and the risky rate as consumption and dividends grow at the same rate
- $(Shrp)$: The Sharpe ratio

Table 8.2 reports the results.[15] The model matches *simultaneously* the moments and, as we see later, it also matches most other features of market dynamics. Kurz et al. (2005a) further observe that the results in Table 8.2 are not due to the particular beliefs used or their parameter values. They are due to fat tails in asset returns, to asymmetry, and to correlation of beliefs as explained in the previous section. Are these three key characteristics supported by the data?

The fact that the distribution of asset returns exhibits fat tails is well documented (e.g., see Fama, 1965, and Shiller, 1981). It is natural to ask where these tails come from. The theory at hand says they come from fat tails in the probability models of agents' beliefs. Correlation of beliefs across agents is documented in sources such as the Blue Chip Financial Forecasts. The evidence in support of the hypothesis that the frequency of bear states is higher than 50% is more complicated.

The hypothesis is supported by the empirical fact that, on average, most above-normal stock returns are realized over *relatively small proportion of time* when asset prices rally (see Shilling, 1992). It is thus reasonable that, on average, the proportion of time when agents expect to make above-normal returns is less than 50%[16]. Additional

[15]For computational procedures to reproduce the simulation results of the Kurz et al. (2005a) model click on "computable models with heterogeneous beliefs" at *www.stanford.edu/~mordecai/.*

[16]Shilling (1992) shows that during the 552 months from January 1946 through December 1991, the mean real annual total return on the Dow Jones Industrials was 6.7%. However, if an investor missed the 50 strongest months, the real mean annual return over the other 502 months was −0.8%. Hence the financial motivation to time the market is very strong, as is the case for the agents in the model.

TABLE 8.2 Simulation Results of Kurz et al. (2005a) (*all moments are annualized*)

Moment	Simulation Results	Empirical Record* (1889–1998)
q	25.54	25.00
σ_q	5.46	7.10
R^q	7.57%	7.00%
σ_R	18.81%	18.00%
r	1.08%	1.00%
σ_r	5.44%	5.70%
e_p	6.49%	6.00%
$\rho(d, R)$	0.21	0.10
Shrp	0.34	0.33

*The main data source for the empirical record is Shiller at *www.econ.vale.edu/~shiller/data.htm.* It was updated by Kurz et al. (2005a, page 23) to 1998. Since the discussion here does not aim to evaluate the precision of the estimates, the numbers in the table were rounded off to indicate orders of magnitude.

indirect support comes from the psychological literature, which suggests that agents place heavier weight on losses than on gains. In the treatment here, agents fear losses at majority of dates since on those dates they place higher probabilities of abnormally lower returns. By the rational belief principle, a higher frequency of bear states implies that in bull states, an agent's intensity of optimism is higher than the intensity of pessimism. This means that the average positive tail in the belief densities is bigger than the average negative tail. That is, the asymmetry hypothesis implies that optimistic agents tend to be intense.

Together with correlation of beliefs across agents, this hypothesis also implies that we should observe periods of high optimism for a majority of agents. Optimism leads to agents' desire to borrow and finance present and future consumption. At such dates the only way for markets to clear is by exhibiting sharp rises in stock prices together with high borrowing rates. Hence, this theory predicts that we should expect to observe rapidly rising stock prices induced by bursts of optimism correlated with high realized growth rates of dividends. The structure of correlation also implies that we should also expect to see crashes induced by correlated pessimistic agents together with low realized growth rates of dividends.

We make one comment with respect to the low riskless rate. Matching many volatility moments *except the riskless rate* depend mostly on intensity and correlation. These moments exhibit relatively low sensitivity to asymmetry. Hence, apart from the riskless rate, many long-term volatility measures are explained by a broad configuration of the intensity parameters and correlation across agents' beliefs. The low riskless rate and a few others require asymmetry.

Why Does the Model of Diverse Beliefs Resolve the Equity Premium Puzzle?

Risk premia are compensations for risk perceived by risk-averse agents. In single-agent models, the market portfolio is identified with a security for which the payoff is aggregate consumption, mostly taken to be exogenous. The Equity Premium Puzzle is thus an observation that the small volatility of aggregate consumption growth cannot justify a 6% equity premium, given the degree of risk aversion. The theory of diverse beliefs offers a resolution of the puzzle by studying optimal behavior and consumption growth rate volatility on the individual, not the aggregate, level.

Any theory of diverse beliefs implies that at each date the risk premium perceived by an agent is subjective. The risk premium required by an investor with a bullish outlook is smaller than the risk premium required by a bear. Hence to resolve the Equity Premium Puzzle a theory must explain why some agents are willing to hold a riskless asset paying a real return of only 1% when the average return on the risky stock is 7%. The 7% return on the stock is entirely explainable by fundamental factors of growth and productivity, together with the added high volatility of returns induced by factors of intensity and correlation of beliefs that generate endogenous uncertainty. The problem is the low riskless rate. Pessimistic agents who aim to preserve capital are willing to earn low return on their investment, and with enough of them around, the riskless rate would indeed fall. But can a desire to preserve capital by those avoiding the risky stock be compatible with fat tails in returns? This is the role of *asymmetry*. Symmetry between bulls and bears generates fat tails only due to intensity and correlation. Agents are intense when they are bulls and correlation causes the majority sentiment to fluctuate. Fat tails then reflect fluctuations of the majority between bull or bear averages. After all, when a majority of agents try to sell or buy the stock, the price fluctuates. But to push the riskless rate down we need the asymmetric persistence of the bear view by those who expect the stock to deliver low excess returns. Expecting low excess returns, they would rather avoid the risky stock and hold the bond at lower return. The fact that bears are in the majority of investors at the majority of dates constitutes the extra factor, which lowers the riskless rate as well.

We turn to the low volatility of aggregate consumption growth. Diverse beliefs cause diverse individual consumption growth rates, even if aggregate consumption is exogenous, which is the case in the models here. This is true not only because of idiosyncratic factors but also because under diverse beliefs markets are inherently incomplete and the representative agent model does not capture the conditions of individual consumers. Hence, volatility of individual consumption growth rates is higher than the volatility of the aggregate rate, an empirical fact supported by household survey data. Since agents' perceived volatility of their own consumption growth is different from the aggregate rate, they do not seek to own a portfolio whose payoff is aggregate consumption. Consequently, we must not focus on the relation between asset returns and aggregate consumption growth but instead on the relation between perceived asset returns and *perceived volatility of individual consumption growth.* The key question is, then, how volatile do individual consumption growth rates need to be to generate an equity premium of 6% and a riskless rate of 1%? The answer is: *not very much.* Relative to their equilibrium, Kurz et al. (2005a) report that although the standard deviation of the aggregate consumption growth rate is 0.03256, the standard deviation of individual consumption growth rates supporting the premium in the simulations is only 0.039, and the required correlation between individual consumption growth rate and the growth rate of dividends is only 0.83 (compared to 1.00 in a representative household model). Both figures are compatible with survey data.

Predictability of Stock Returns

The problem of predictability of risky returns generated a large literature in empirical finance (e.g., Fama and French, 1988a, 1998b; Poterba and Summers, 1988; Campbell and Shiller, 1988; and Paye and Timmermann, 2003). This debate originates in the theoretical observation that under risk aversion asset prices and returns are not martingales and contain a predictable component. In this context Kurz et al. (2005a) use the model associated with Table 8.2 to generate simulated data with which they examine the following: (1) variance ratio statistic; (2) autocorrelation of returns and of price/dividend ratios; and (3) predictive power of the dividend yield. They then apply to simulated data the standard tests used for market data, and we report their results in the following subsections. Recalling that v_t is the growth rate of dividends and q_t is the price dividend ratio, the standard notation used in this literature is to let $\varrho_t = \log[\frac{(q_t+1)e^{v_t}}{q_{t-1}}]$ be the log of gross one year stock return, $\varrho_t^k = \sum_{i=0}^{k-1} \varrho_{t-i}$ be the cumulative log-return of k-year length from $t-k+1$ to t, and $\varrho_{t+k}^k = \sum_{j=1}^{k} \varrho_{t+j}$ be the cumulative log-return over a k-year horizon from $t+1$ to $t+k$.

Variance Ratio Test The variance-ratio is $VR(k) = \frac{var(\varrho_t^k)}{(k\, var(\varrho_t))}$. If returns are uncorrelated, this ratio converges to 1 as k rises. If returns are negatively autocorrelated at some lags, the ratio is less than one. Kurz et al. (2005a) show there exists a significant higher-order autocorrelation in simulated stock returns and hence a long-run predictability that is consistent with U.S. data on stock returns, as in Poterba and Summers (1988). In Table 8.3 Kurz et al. (2005a) report the computed values of the ratios for $k = 1, 2, \ldots, 10$ and compare them with the ratios in the empirical record reported by Poterba and Summers (1988, Table 8.2, line 3) for $k = 1, 2, \ldots, 8$. The model's prediction is close to the U.S. empirical record.

TABLE 8.3 Variance Ratios for NYSE 1926 to 1985

k	1	2	3	4	5	6	7	8	9	10
$VR(k)$	1.00	0.85	0.73	0.64	0.57	0.51	0.46	0.41	0.38	0.34
U.S.	1.00	0.96	0.84	0.75	0.64	0.52	0.40	0.35	—	—

The Autocorrelation of Log-Returns and Price–Dividend Ratios In Table 8.4 we report the Kurz et al. (2005a) autocorrelation function of log annual returns. The model

TABLE 8.4 Autocorrelation of Log-Returns

corr(ϱ_t, ϱ_{t-k})	Model	Empirical Record
$i = 1$	−0.154	0.070
$i = 2$	−0.094	−0.170
$i = 3$	−0.069	−0.050
$i = 4$	−0.035	−0.110
$i = 5$	−0.040	−0.040

TABLE 8.5 Autocorrelation of Price–Dividend Ratio

corr(q_t, q_{t-k})	Model	Empirical Record
$i = 1$	0.695	0.700
$i = 2$	0.485	0.500
$i = 3$	0.336	0.450
$i = 4$	0.232	0.430
$i = 5$	0.149	0.400

predicts negatively autocorrelated returns at all lags. This implies a long horizon mean reversion of the kind documented by Poterba and Summers (1988), Fama and French (1998a), and Campbell and Shiller (1988). Thus, apart from the very short returns that exhibit positive autocorrelation, the model reproduces the empirical record.

In Table 8.5 we report the autocorrelation function of the price-dividend ratio reported by Kurz et al. (2005a). The table shows the model generates a highly autocorrelated price/dividend ratio, which matches reasonably well the behavior of U.S. stock market data. The empirical record in Tables 8.4 and 8.5 is for NYSE data for 1926–1995 as reported in Barberis et al. (2001).

Dividend Yield as a Predictor of Future Stock Returns The papers cited previously show that the price–dividend ratio is the best explanatory variable of long returns. To test this fact Kurz et al. (2005a) consider the following regression model:

$$\varrho^k_{t+k} = \zeta_k + \eta_k(e^{v_t}/q_{t-1}) + \vartheta_{t,k} \qquad \textbf{(8.31)}$$

e^{v_t}/q_{t-1} is the dividend yield since it is the ratio between dividend paid at t and the stock price at $t-1$.

Fama and French (1988b) report that the ability of the dividend yield to forecast stock returns, measured by regression coefficient R^2 of Eq. 8.31, increases with the return horizon. Kurz et al. (2005a) find that the model captures the main features of the empirical evidence as reported in Table 8.6.

TABLE 8.6 The Behavior of the Regression Slopes in Eq. 8.31

Time Horizon	Model		Empirical Record	
k	η_k	R^2	η_k	R^2
1	5.03	0.08	5.32	0.07
2	8.66	0.14	9.08	0.11
3	11.16	0.18	11.73	0.15
4	13.10	0.21	13.44	0.17

To conclude the discussion of predictability, observe that the empirical evidence reported by Fama and French (1998a, 1998b), Campbell and Shiller (1988), Poterba and Summers (1998), and others is consistent with asset price theories in which time-varying expected returns generate predictable, mean-reverting components of prices (see Summers, 1986). The important question left unresolved by these papers is, what drives the predictability of returns implied by such mean-reverting components of prices? Part of the answer is the persistence of the dividend growth rate. Kurz et al. (2005a) offer a second and stronger persistent mechanism. It shows that these results are primarily driven by the dynamics of market state of beliefs, which exhibit correlation across agents and persistence over time. Agents go through bull and bear states, causing their perception of risk to change and expected returns to vary over time. Equilibrium asset prices depend on states of belief that then exhibit memory and mean reversion. Hence returns exhibit these same properties.

GARCH Behavior of the Price–Dividend Ratio and of the Risky Returns

Stochastic volatility in asset prices and returns is well documented (e.g., Bollerslev, Engle, and Nelson, 1994; Brock and LeBaron, 1996). In partial equilibrium finance it is virtually standard to model asset prices by stochastic differential equations, *assuming* an exogenously driven stochastic *volatility*. But where does stochastic volatility come from? Dividends certainly do not exhibit stochastic volatility. We now show that models with diverse beliefs *can explain why asset prices and returns exhibit stochastic volatility*.

To formally test the GARCH property of the price–dividend ratio and of the risky returns, Kurz et al. (2005a) use the 20,000 simulated observations noted in the "Equity Premium Puzzle" section. With that data they estimate the following econometric model of the dynamics of the log of the price–dividend ratio:

$$\begin{aligned} \log(q_{t+1}) &= \kappa^q + \mu^q \log(q_t) + \zeta^q_{t+1} \\ \zeta^q_t &\sim N(0, h^q_t) \\ h^q_t &= \xi^q_0 + \xi^q_1(\zeta^q_{t-1})^2 + v^q_1 h^q_{t-1} \end{aligned} \tag{8.32a}$$

Since the price–dividend ratio is postulated to be an AR(1) process, the process in Eq. (8.32a) is GARCH(1,1). Similarly, for the risky rates of return, they postulated the model

$$\begin{aligned} \varrho_{t+1} &= \kappa^{\varrho} + \mu^{\varrho} \log(q_{t+1}) + \varsigma^{\varrho}_{t+1} \\ \varsigma^{\varrho}_{t} &\sim N(0, h^{\varrho}_{t}) \\ h^{\varrho}_{t} &= \xi^{\varrho}_{0} + \xi^{\varrho}_{1}(\varsigma^{\varrho}_{t-1})^2 + \nu^{\varrho}_{1} h^{\varrho}_{t-1} \end{aligned} \tag{8.32b}$$

For a specification of Eqs. 8.32a and 8.32b, they also tested ARCH(1) and GARCH(2,1) but concluded that the proposed GARCH(1,1) describes best the behavior of the data. Due to the large sample they ignore standard errors and report that the estimated model for the log of the price–dividend ratio satisfies the GARCH(1,1) specification:

$$\begin{aligned} \log(q_{t+1}) &= 0.99001 + 0.69384 \log(q_t) + \varsigma^{q}_{t+1} \\ \varsigma^{q}_{t} &\sim N(0, h^{q}_{t}) \\ h^{q}_{t} &= 0.00592 + 0.02370(\varsigma^{q}_{t-1})^2 + 0.73920 h^{q}_{t-1}, R^2 = 0.481 \end{aligned}$$

For risky rates of return, the estimated model satisfies the GARCH(1,1) specification

$$\begin{aligned} \varrho_{t} &= 1.13561 - 0.33355 \log(q_t) + \varsigma^{\varrho}_{t} \\ \varsigma^{\varrho}_{t} &\sim N(0, h^{\varrho}_{t}) \\ h^{\varrho}_{t} &= 0.00505 + 0.01714(\varsigma^{\varrho}_{t-1})^2 + 0.77596 h^{\varrho}_{t-1}, \quad R^2 = 0.180 \end{aligned}$$

To explain we observe that stochastic volatility is a direct consequence of the dynamics of beliefs, defined by (z^1_t, z^2_t) in Kurz et al. (2005a). Persistence of beliefs and correlation across agents introduce these patterns into prices and returns. When agents disagree (i.e., $z^1_t z^2_t < 0$), they offset the demands of each other and as that pattern persists, prices do not need to change by very much for markets to clear. During such periods prices exhibit low volatility: persistence of belief states induce persistence of low volatility. When agents agree (i.e., $z^1_t z^2_t > 0$) they compete for the same assets and prices are determined by difference in belief intensities. Changes in the levels of bull or bear states generate high volatility in asset prices and returns. Persistence of beliefs cause such high-volatility regimes to exhibit persistence. Market volatility is then time dependent and has a predictable component as in Eqs. 8.32a and 8.32b.

The virtue of the above argument is that it explains stochastic volatility *as an endogenous consequence of equilibrium dynamics*. Some "fundamental" shocks (i.e., an oil shock) surely cause market volatility, but it has been empirically established that market volatility cannot be explained *consistently* by "fundamental" exogenous shocks (e.g., Schwert, 1989; Pesaran and Timmermann, 1995; and Beltratti and Morana, 2006). The Kurz et al. (2005a) explanation of stochastic volatility is thus consistent with the empirical evidence.

8.4.3. Volatility of Foreign Exchange Rates and the Forward Discount Bias

The relevance of foreign exchange markets to our discussion in this chapter is motivated by the following problem. Estimate a regression of the form

$$\frac{ex_{t+1} - ex_t}{ex_t} = c + \zeta(r_t^D - r_t^F) + \epsilon_{t+1} \tag{8.33}$$

where $(ex_{t+1} - ex_t)$ is the change of the exchange rate between t and $t + 1$ and $(r_t^D - r_t^F)$ is the difference between the short-term nominal interest rates in the domestic and the foreign economies. Under rational expectations $(r_t^D - r_t^F)$ is an unbiased predictor of $(ex_{t+1} - ex_t)$. It is motivated by a standard arbitrage argument: If there is a differential in nominal rates, agents can borrow in one country and invest in the other and gain from the difference if the exchange rate does not move against them by the amount of the differential. In a no-arbitrage REE, a rationally expected change in the exchange rate must then be equal to the interest rate differential. This means that apart from a technical correction for risk aversion, the parameter ζ should be close to 1. In 75 empirical studies ζ was estimated to be significantly less than 1 and in many studies it was estimated to be negative (see Froot and Frankel, 1989; Frankel and Rose, 1995; and Engel, 1996, for an extensive survey).

The failure of ζ to exhibit estimated values close to 1 is known as the *forward discount bias* in foreign exchange markets. The empirical fact is that exchange rates are far more volatile than can be explained by differentials in nominal interest rates or inflation rates between countries. But changes in foreign exchange rates are not predictable and interest rate differentials account only for a small fraction of the movements in foreign exchange rates. However, it is not surprising that this lack of predictability decreases with the length of time involved. That is, long-run differentials in nominal interest rates do exhibit better predictive power of long-run movements in foreign exchange rates, since long-run differentials in nominal rates reflect differentials in inflation rates. Since the problem at hand is the nature of market expectations and exchange rate volatility, it is a natural for us to consider it here, and the model of diverse belief is an obvious candidate to be used to solve the problem.

Applying the rational belief theory to this problem, Kurz (1997b) and Black (1997), (2005) developed a model that is similar to the Kurz and Motolese (2001) model except for treating the second agent as a second country and adding two nominal debt instruments. A similar model was also reformulated by Kurz and Motolese (2001). Limitation of space makes it impossible to review all technical details of these models here. Instead, we outline the key points of the model construction and note the results. Hence, the central model construction elements in these papers are as follows:

- Consider the first agent as the "domestic United States," which is the home country, and the second agent as a "foreign economy"
- Introduce a second shock that is associated with productivity in the foreign economy and is different from the first shock defined for productivity in the domestic economy

- Introduce a monetary system for both countries and a second currency
- Introduce two nominal interest rates and two different monetary policies
- There are the two standard financial assets: (1) ownership shares of a domestic firm with stochastic dividend whose stock trades freely in both currencies across the countries, and (2) a zero net supply riskless bond that pays a unit of consumption and that trades in both currencies across the countries
- Introduce a simple production structure for the foreign economy

Note that these models do not aim to simulate the United States or world economies; they merely aim to explain via simulations why a model with diverse beliefs implies $\zeta < 1$. And indeed, all models produce estimated parameters ζ, that are significantly less than 1: In the *rational belief equilibrium* of Kurz (1997b) the estimated ζ is around 0.25, in Black (1997, 2005) it is around 0.15, and in Kurz and Motolese (2001) it is around 0.45. More realistic results could be obtained by formulating more realistic models, but the key result $\zeta < 1$ is virtually independent of the model formulation. We now provide an explanation for this strong conclusion.

Why do diverse beliefs predict that ζ is less than 1? If $\zeta < 1$ in an REE, agents can make an *expectational* arbitrage: They can borrow today in one currency, invest in the other, and *expect* that the net return on their investment *next period* will be larger than the depreciation of the currency. In such an equilibrium all agents hold the same self-fulfilling expectations; the *expectational* arbitrage becomes a *real* arbitrage and consequently this implies that $\zeta < 1$ cannot hold in equilibrium.

In a world with diverse beliefs, equilibrium exchange rate depends on the distribution of beliefs and hence exchange rates exhibit excess volatility, reflecting the variability of investors' beliefs. Indeed, volatility of foreign exchange rates is dominated by endogenous uncertainty. The implication is that *regardless of the information today*, to forecast future exchange rates agents must forecast future market states of belief, rendering exchange rates virtually unpredictable. Hence, if a condition of differential nominal interest rates across countries arises, it can never be the only factor that will determine the exchange rate next period. With risk-averse agents who are unable to predict the exchange rate, a condition of differential interest rates will not generate the beliefs of traders that the exchange rate will, in fact, adjust. Failing to expect the exchange rate to adjust, they will not undertake such arbitrage and the exchange rate will, in fact, not adjust. This mechanism ensures that a differential of nominal interest rates between the two countries is not an unbiased estimate of the rate of depreciation of the exchange rate one period later, hence $\zeta < 1$. This reasoning does not hold in the long run since a long-term differential of nominal interest rates will persuade the markets that the exchange rate must adjust in the long run and this will persuade them to engage in such arbitrage.

8.4.4. Macroeconomic Applications

Although there is a wide range of potential applications in macroeconomics, so far only limited questions have been studied with the model of diverse beliefs. Motolese (2001, 2003) shows that in an economy with diverse beliefs, money is not neutral. To see why, it is important to observe that before *rational expectations* the case for money neutrality was based on the quantity theory of money. The main contribution of Lucas (1972) was to show that money neutrality can be proved only by an exploration of the structure of expectations. In a model with heterogeneous beliefs, agents hold diverse beliefs about the relative effects of productivity growth and money shocks; hence they hold diverse beliefs about future inflation. With diverse expectations money cannot be neutral.

Kurz et al. (2005b) is a comprehensive study of the efficacy of monetary policy in an economy with diverse beliefs. The authors show that diverse beliefs constitute an important propagation mechanism of fluctuations, money nonneutrality, and efficacy of monetary policy. Since expectations affect demand, the theory shows that economic fluctuations are driven mostly by varying demand, not supply shocks. Using a competitive model with flexible prices in which agents hold rational beliefs, the authors arrive at six conclusions:

1. The model economy replicates well the empirical record of fluctuations in the United States.
2. Under monetary rules without discretion, monetary policy has a strong stabilization effect, and an aggressive anti-inflationary policy can reduce inflation volatility to zero.
3. The statistical Phillips Curve changes substantially with policy instruments, and activist policy rules render it vertical.
4. Although prices are flexible, money shocks result in less than proportional changes in inflation; hence the aggregate price level is "sticky" with respect to money shocks.
5. Discretion in monetary policy adds a random element to policy and increases volatility. The impact of discretion on the efficacy of policy depends on the structure of market beliefs about future discretionary decisions. The paper studies two rationalizable beliefs. In one case, market beliefs weaken the effect of policy; in the second, beliefs bolster policy outcomes. Therefore, in this case, discretion is a desirable attribute of the policy rule. That is, social gains from discretion arise only under special structures of belief of the private sector about future bank discretionary acts, and such requirement complicates the bank's problem. Hence, the weight of the argument leads Kurz et al. (2005b) to conclude that a bank's *policy should be transparent and abandon discretion except for rare and unusual circumstances*. This analysis is in contrast to the recent literature initiated by Morris and Shin (2005) and others who suggest that due to asymmetric private information, central bank transparency has inherent *cost of failing to retrieve useful private information* by the bank. We have rejected the applicability of the private information model for the study of economic aggregates such as interest rates, inflation rate, or GDP growth. Hence, the Morris and Shin (2005) model does not address the real problem associated with the objective of central bank transparency, which is the coordination of expectations.
6. One implication of the model suggests that the present-day policy is only mildly activist and aims mostly to target inflation.

8.5. CONCLUSION AND OPEN PROBLEMS

Rational expectations and irrational behavior are extreme hypotheses; with REE one cannot explain the observed data on market dynamics and with irrational behavior one can prove anything. We highlight here the merit of an intermediate concept of belief rationality that emerges from the fact that the economy is a nonstationary system with time-varying structure. This prevents agents from ever learning true structural relations and probability laws. All they learn are the empirical frequencies from which emerge a common knowledge of a stationary probability reflecting the long-term dynamics. Belief rationality requires agents to hold only beliefs that are not contradicted by the empirical evidence. But since it is irrational to believe in a fixed deviation from the stationary probability, such belief rationality implies belief dynamics: Individual beliefs must be time varying, and correlation across agents generates a new aggregate force in market dynamics, which is the dynamics of market belief.

The main observation made in this chapter is that the dynamics of market beliefs are a central market force that is as important to asset pricing and allocation as the dynamics of productivity or public policy. Indeed, the dynamics of market beliefs explain well the four recessions Samuelson noted that the market predicted but that did not happen. It shows that a rational market makes forecasting mistakes and rational investors are not infallible. They may use wrong forecasting models. Once we recognize that being rational and being wrong are not incompatible and that no psychological impulses are needed for this proposition, we are open to a new paradigm of market dynamics. This paradigm provides a coherent explanation to most dynamical phenomena of interest as outlined in this chapter. We thus sum up our five central conclusions:

1. Diverse beliefs without any private information are an empirical fact, and such diversity provides a strong motive to trade assets and hedge subjectively perceived risks.
2. Financial markets are the great arena for agents to trade differences in beliefs.
3. The dynamics of market beliefs are a central component of asset price volatility, and this component of risk has been named *endogenous uncertainty*. Market belief is observable.
4. Asset markets exhibit large excess volatility of prices, returns, and high volume of trade due to the dynamics of beliefs.
5. Risk premia reflect the added market risk due to the dynamics of beliefs and in some markets the component of risk premia due to the dynamics of market beliefs is very significant.

Important problems that we have not discussed are still open, some of which are being researched at this time. Five examples are as follows:

1. *Pareto optimality*. The concept of ex-ante Pareto optimum is not a satisfactory concept for a market with diverse beliefs. To attain any Pareto improvement, all agents must believe it is an improvement, and that is not likely. Hence, most stabilization policies would not be Pareto improving. Following the idea of ex-post Pareto optimality (e.g., Starr, 1973, and Hammond, 1981), progress on this issue was made

by Nielsen (2003, 2006), who argued that a currency union is superior to multiple currencies since a union would eliminate endogenous uncertainty inherent in foreign exchange rates.

2. *Stabilization policy*. When the problem of Pareto optimality is resolved, the door will be open to a study of the desirability of stabilization public policies. Some start has been made by Kurz et al. (2005b) regarding stabilizing monetary policy. But the question is broader. Should the Fed target the stock market? Should countries cooperate to avoid an international financial crisis? What is the role of an international convention regarding bank reserve requirements? Under REE these types of questions are set aside since it is often argued that the market solution is best and no cooperative policy is needed. In a world of diverse beliefs, this is not true and the question is open.
3. *Continuous time reformulation*. A continuous time reformulation of the RB theory would open the door to a study of the decomposition of risk into fundamental and endogenous components. With such formulation available, we can formulate the decomposition of the values of derivative securities, using Black Scholes, into the fundamental and endogenous components. Such a decomposition is likely to provide an explanation to the Smile Curves in derivative pricing.
4. *Destabilizing speculation of futures markets*. Could the opening of a future's market increase the volatility of a spot market? This is an old question that has not been fully clarified. Our conjecture is that a proper formulation of the problem will show that if margin requirements and leverage conditions are sufficiently relaxed in a futures market, its opening could give rise to endogenous uncertainty, which cannot arise in the spot market if storage cost are high enough. This could increase the volatility of a spot market.
5. *Volume of trade and speculation*. Markets with diverse beliefs are the natural arenas for agents to trade differences in their beliefs, and we have reviewed recent progress made by Wu and Guo on this problem. However, the problem of speculation needs to be solved for economies with risk aversion. Also, a significant amount of empirical work has been done on patterns of the volume of trade, but much remains to be explained with formal models.

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